HOMEWORK 5.

Solutions

1

a) By Theorem 13.9 in course notes, the distribution of $X-Y$ for independent X and Y depends only on the distributions of X and Y. Since X and X' have same distributions, this fact implies that $X-X'$ and $X'-X$ have same distributions, i.e. $X-X'$ is symmetric.

b) $$\varphi_{X^s}(t) = \mathbb{E}e^{it(X-X')} = \mathbb{E}e^{itX}\cdot\mathbb{E}e^{-itX'} \quad \text{by independence}$$
$$= \mathbb{E}e^{itX}\cdot\mathbb{E}e^{-itX} \quad \text{by identical distribution}$$
$$= \varphi_X(t)\cdot\varphi_X(-t)$$
$$= \varphi_X(t)\cdot\overline{\varphi_X(t)} = |\varphi_X(t)|^2.$$

c) $$\varphi_{(X+Y)^s}(t) = |\varphi_{X+Y}(t)|^2 \quad \text{by (b)}$$
$$= |\varphi_X(t)\,\varphi_Y(t)|^2 \quad \text{by independence}$$
$$= |\varphi_X(t)|^2\cdot|\varphi_Y(t)|^2$$
$$= \varphi_{X^s}(t)\cdot\varphi_{Y^s}(t) \quad \text{by (b)}$$
$$= \varphi_{X^s+Y^s}(t) \quad \text{by independence.}$$

Hence, by the Uniqueness Theorem (Corollary 22.5), $(X+Y)^s = X^s + Y^s$ in distribution.

2

$X - Y$ has the same distribution as X^s. By Problem 1,

$$\varphi_{X-Y}(t) = |\varphi_X(t)|^2$$

~~and~~ so, in particular, it is non-negative.

The random variable U uniformly distributed on $[-1,1]$ has ch. f.

$$\varphi_U(t) = \frac{\sin t}{t}$$

(see Section 21.1 of Course Notes). This function can have negative values. Hence $X-Y$ can not be distributed identically with U.

3

Suppose that $|X_k| \le K$ for all k.

Check Lyapunov's Condition with $p=4$ (for example):

$$\left(\sum_1^n \mathbb{E}|X_k^4| \right)^{1/4} \le \left(\sum_1^n K^2 \mathbb{E} X_k^2 \right)^{1/4} = K^{1/2} s_n^{1/2} = \bar{o}(s_n) \quad \text{as } n \to \infty,$$

because $s_n \to \infty$ as $n \to \infty$. Therefore, CLT holds.

4

X_k has mean zero and variance $\sigma_k^2 = 15 \cdot \left(\frac{1}{16}\right)^k$. Hence

S_n has mean zero and variance $s_n^2 = 15 \sum_{k=1}^n \left(\frac{1}{16}\right)^k = 1 - \left(\frac{1}{16}\right)^n \to 1$ as $n \to \infty$.

However, note that $\mathbb{P}(|S_n| \le \frac{1}{2}) = 0$. Since $s_n \to 1$, it follows that

$$(*) \qquad \mathbb{P}\left(\left| \frac{S_n}{s_n} \right| \le \frac{1}{4} \right) = 0 \quad \text{for all sufficienly large } n.$$

Moreover, for a standard normal random variable,

$$(**) \qquad \mathbb{P}(|N| \le \tfrac{1}{4}) \text{ is some constant } > 0.$$

Then $(*)$ and $(**)$ show that $S_n/s_n \not\to N$ in distribution. QED

5

Statement of CLT: Let $S_n = X_1 + \dots + X_n$, $s_n^2 = \mathrm{var}(S_n)$.

Then
$$\frac{S_n}{s_n} \to N(0,1) \quad \text{in distribution, as } n \to \infty.$$

We check Lyapunov's condition. Let $p > 2$, p even. ~~an It suffice~~

~~In view of~~ It suffices to prove that
$$\beta_n := \frac{\left(\sum_{k=1}^{n} \mathbb{E} X_k^p\right)^{1/p}}{\left(\sum_{k=1}^{n} \mathbb{E} X_k^2\right)^{1/2}} \to 0 \quad \text{as } n \to \infty.$$

Note that $\mathbb{E} X_k^p = k^{\alpha p}$. Then
$$\left(\sum_{k=1}^{n} \mathbb{E} X_k^p\right)^{1/p} = \left(\sum_{k=1}^{n} k^{\alpha p}\right)^{1/p} \le C_{\alpha,p}\left(\int_1^n x^{\alpha p}\,dx\right)^{1/p} \le C'_{\alpha,p}\left(n^{\alpha p+1}\right)^{1/p}$$
$$\le C''_{\alpha,p}\, n^{\alpha + 1/p}.$$

Similarly,
$$\left(\sum_{k=1}^{n} \mathbb{E} X_k^2\right)^{1/2} = \left(\sum_{k=1}^{n} k^{2\alpha}\right)^{1/2} \le \begin{cases} C''_{\alpha,p}\, n^{\alpha+1/2} & \text{if } \alpha > -1/2 \\ \dfrac{C''_{\alpha,p}}{\sqrt{\log n}} & \text{if } \alpha = 1/2. \end{cases}$$

Hence
$$\beta_n \le \left.\begin{cases} C'''_{p,\alpha} \cdot n^{\frac{1}{p} - \frac{1}{2}} & \text{if } \alpha > -\frac{1}{2} \\ C'''_{p} \cdot \dfrac{n^{\frac{1}{p}-\frac{1}{2}}}{\sqrt{\log n}} & \text{if } \alpha = -\frac{1}{2}. \end{cases}\right\} \to 0 \quad \text{as } n \to \infty.$$

QED.