**Homework 5**

Probability Theory (MATH 235B, Winter 2008)

**1.** A *symmetrization* of a random variable $X$ is a random variable $X^s = X - X'$ where $X'$ is an independent copy of $X$.

(a) Prove that $X^s$ is a symmetric random variable.

(b) Express the characteristic function of $X^s$ in terms of the characteristic function of $X$.

(c) Let $X$ and $Y$ be independent random variables. Show that

$$(X + Y)^s = X^s + Y^s$$

in distribution (i.e. the left and right hand sides have the same distribution).

**2.** Show that there do not exist independent, identically distributed random variables $X$ and $Y$ such that $X - Y$ is distributed uniformly in $[-1, 1]$.

**3.** Prove the following version of the Central Limit Theorem. Let $X_1, X_2, \ldots$ be independent and uniformly bounded random variables with means 0. Let $S_n = X_1 + \ldots X_n$. If the variance $s_n^2$ of $S_n$ goes to infinity then

$$\frac{S_n}{s_n} \to N \quad \text{in distribution,}$$

where $N$ is the standard normal random variable. *(Hint: use Lyapunov's Condition).*

**4.** Let $X_1, X_2, \ldots$ be independent random variables such that

$$\mathbb{P}\Big(X_k = \frac{\sqrt{15}}{4^k}\Big) = \mathbb{P}\Big(X_k = -\frac{\sqrt{15}}{4^k}\Big) = \frac{1}{2}.$$

Show that these random variables do not obey the usual Central Limit Theorem. *Hint: what is* $\mathbb{P}(|S_n/s_n| \le 1/10)$*?*

**5.** Let $X_1, X_2, \ldots$ be independent random variables such that

$$\mathbb{P}(X_k = k^\alpha) = \mathbb{P}(X_k = -k^\alpha) = \frac{1}{2},$$

where $\alpha \ge -1/2$. Prove state the Central Limit Theorem for these random variables.