1. We use the convexity of $x \mapsto e^{\lambda x}$ and the fact that $|x| \leq 1$.

For $|x| \leq 1$, $\lambda x \in [-\lambda, \lambda]$, write $\lambda x$ as the convex combination of $-\lambda$ and $\lambda$:

$$
\lambda x = p\lambda + (1-p)\lambda
$$

Solving for $p$ gives

$$
p = \frac{1+x}{2}, \quad 1-p = \frac{1-x}{2}.
$$

By convexity,

$$
e^{\lambda x} \leq e^{\lambda \frac{1+x}{2}} + (1-p)e^{\lambda \frac{1-x}{2}} = \left(\frac{1+x}{2}\right)e^\lambda + \left(\frac{1-x}{2}\right)e^\lambda = ch\lambda + xsh\lambda.
$$

Since $\mathbb{E}X = 0$, we obtain

$$
\mathbb{E} e^{\lambda X} \leq ch\lambda.
$$

It remains to note that $ch\lambda \leq e^{\lambda/2}$ to finish the proof.

2. Let $M > 0$. The event $\mathbb{E}\{\text{lim sup } S_n/n > M\}$ is clearly in the tail $\sigma$-algebra generated by the r.v.'s $(X_n)$, because the values of any finite number of these r.v.'s do not change whether $\mathbb{E}$ holds or not.

By Kolmogorov's 0-1 law, $P(\mathbb{E}^c) = 0$ or 1.

Let us show that $P(\mathbb{E}^c) > 0$.

By CLT, $P(S_n/n > M) \to P(N > M) = \mathbb{P}(M) > 0$,

where $N$ is a normal random variable with the same variance as $X_k$. 


Using the continuity property of probability measures (Cor. 1.4), we have

\[ P(\xi) \geq \text{lim sup } P\left( \frac{S_n}{\sqrt{n}} > M \right) = P(M) > 0. \]

Therefore, \( P(\xi_N) = 1. \)

Since \( M \) is arbitrary,

\[ P \left( \text{lim sup } \frac{S_n}{\sqrt{n}} = \infty \right) = P \left( \bigcap_{M=1}^{\infty} E_M \right) = 1. \quad \text{(Q.E.D.)} \]

\[ X_n = g_n + h_n \]

where \( g_n \) are "good" random variables, like i.i.d. N(0,1),

\( h_n \) are "bad" random variables, like i.i.d. Cauchy

\[ i.e. \quad \text{without 2-nd moment,} \quad \text{(no 2nd moment)} \]

\( E_n \to 0 \) very fast.
We have: \( \forall \varepsilon > 0, \)

\[
P(\left| Y_n - a \right| > \varepsilon) \to 0 \quad \text{as} \ n \to \infty
\]

Also, since \( X_n \to X \) in distribution, \((X_n)\) is tight, hence:

\[
\sup_n P(\left| X_n \right| > M) \to 0 \quad \text{as} \ M \to \infty
\]

Writing \( aX_n = Y_nX_n + (a - Y_n)X_n, \) and estimating

\[
aX_n \leq Y_nX_n + |a - Y_n|X_n,
\]

we have: \( \forall x \) point of continuity of the d.f. of \( X, \)

\[
P(\left| X_n \right| \leq x) = P(Y_nX_n \leq x, |a - Y_n| \leq \varepsilon, |X_n| \leq M) + P(|a - Y_n| > \varepsilon) + P(|X_n| > M)
\]

\[
\leq P(aX_n \leq x + \varepsilon M) + P(|a - Y_n| > \varepsilon) + P(|X_n| > M).
\]

Using the Skorokhod's Representation Theorem, one easily shows that \( X_n \to X \) implies \( aX_n \to aX \) in distribution.

Hence, \( P(aX_n \leq x + \varepsilon M) \to P(aX \leq x + \varepsilon M). \)

We have proved that

\[
\limsup_n P(\left| Y_nX_n \right| \leq x) \leq P(aX \leq x + \varepsilon M) + \limsup_n P(|a - Y_n| > \varepsilon) + \limsup_n P(\left| X_n \right| > M)
\]

The second term in RHS equals 0 by (8) for every \( \varepsilon > 0. \)

Now choose \( M = \sqrt{\varepsilon} \) and let \( \varepsilon \to 0. \) By (**), the third term in RHS \( \to 0, \) and, by continuity, \( EM \to 0, \) so by continuity

\[
\limsup_n P(\left| Y_nX_n \right| \leq x) \leq P(aX \leq x).
\]

The converse inequality is similarly proved (for \( \liminf \)).

\( \Box \)
Write the ratio as

$$R_n = \frac{\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_k}{\left(\frac{1}{n} \sum_{i=1}^{n} X_k^2\right)^{1/2}}$$

With $\sigma^2 = \mathbb{E} X_k^2$, the Strong CLT gives

$$\frac{1}{n} \sum_{i=1}^{n} X_k^2 \to \mathbb{E} X_k^2 = \sigma^2, \quad \text{a.s.}$$

while CLT gives

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_k \to N \quad \text{in distribution.}$$

where $N$ is $N(0, \sigma^2)$.

Now, (4) clearly yields

$$\frac{1}{\left(\frac{1}{n} \sum_{i=1}^{n} X_k^2\right)^{1/2}} \to \frac{1}{\sigma} \quad \text{a.s.}$$

By the result in Problem 3, we get

$$R_n \to \frac{N}{\sigma} \quad \text{in distribution.}$$

Note that $\frac{N}{\sigma}$ is $N(0,1)$. Q.E.D.