HOMEWORK 6 Solutions

①

We use the convexity of $x \mapsto e^{\lambda x}$ and the fact that $|X| \le 1$.

For $|x| \le 1$, $\lambda x \in [-\lambda, \lambda]$. Write λx as the convex combination of $-\lambda$ and λ:

$$\lambda x = p\lambda + (1-p)\lambda$$

Solving for p gives

$$p = \frac{1+x}{2}, \qquad 1-p = \frac{1-x}{2}.$$

By convexity, $$e^{\lambda x} \le p e^{\lambda} + (1-p)e^{-\lambda} = \left(\frac{1+x}{2}\right)e^{\lambda} + \left(\frac{1-x}{2}\right)e^{-\lambda} = ch\lambda + x\, sh\lambda.$$

Since $\mathbb{E}X = 0$, we obtain

$$\mathbb{E}e^{\lambda X} \le ch\lambda.$$

It remains to note that $ch\lambda \le e^{\lambda^2/2}$ to finish the proof.

②

Let $M > 0$. The event $\mathcal{E}_M = \{\limsup S_n/\sqrt{n} > M\}$ is clearly in the tail σ-algebra generated by the r.v's (X_k), because the values of any finite number of these r.v's do not change whether $\mathcal{E}$ holds or not.

By Kolmogorov's 0-1 law, $\mathbb{P}(\mathcal{E}_M) = 0$ or 1.

Let us show that $\mathbb{P}(\mathcal{E}_M) > 0$.

By CLT, $\mathbb{P}(S_n/\sqrt{n} > M) \to \mathbb{P}(N > M) =: p(M) > 0$.

where N is a normal random variable with the same variance as X_k.

Using the continuity property of probability measures (Cor. 1.4), we have

$$\mathbb{P}(\mathcal{E}_M) \geq \limsup \mathbb{P}(S_n/\sqrt{n} > M) = p(M) > 0.$$

Therefore, $\mathbb{P}(\mathcal{E}_M) = 1$.

Since M is arbitrary,

$$\mathbb{P}\left(\limsup \frac{S_n}{\sqrt{n}} = \infty\right) = \mathbb{P}\left(\bigcap_{M=1}^{\infty} \mathcal{E}_M\right) = 1.$$

QED.

5

$X_n = g_n + \varepsilon_n h_n$

where g_n are "good" random variables, like i.i.d. $N(0,1)$

h_n are "bad" random variables, ~~like i.i.d. Cauchy~~ ~~(no 2-nd moment~~

i.e. without 2-nd moment,

$\varepsilon_n \to 0$ very fast.

We have: $\forall \varepsilon > 0$,

(*) $\quad \mathbb{P}(|Y_n - a| > \varepsilon) \to 0 \quad$ as $n \to \infty$

Also, since $X_n \to X$ in distribution, (X_n) is tight, hence:

(**) $\quad \sup_n \mathbb{P}(|X_n| > M) \to 0 \quad$ as $M \to \infty$

Writing $aX_n = Y_n X_n + (a - Y_n) X_n$, and estimating

$$a X_n \leq Y_n X_n + |a - Y_n| \cdot |X_n|,$$

we have: $\forall x$ point of continuity of the d.f. of X:

$$\mathbb{P}(Y_n X_n \leq x) = \mathbb{P}(Y_n X_n \leq x,\ |a - Y_n| \leq \varepsilon,\ |X_n| \leq M) + \mathbb{P}(|a - Y_n| > \varepsilon) + \mathbb{P}(|X_n| > M)$$

$$\leq \mathbb{P}(aX_n \leq x + \varepsilon M) + \mathbb{P}(|a - Y_n| > \varepsilon) + \mathbb{P}(|X_n| > M).$$

Using the Skorokhod's Representation Theorem, ~~with the~~ one easily shows that $X_n \to X$ implies $aX_n \to aX$ (in distribution)

Hence, $\mathbb{P}(aX_n \leq x + \varepsilon M) \to \mathbb{P}(aX \leq x + \varepsilon M)$.

We have proved that

$$\limsup_n \mathbb{P}(Y_n X_n \leq x) \leq \mathbb{P}(aX \leq x + \varepsilon M) + \limsup_n \mathbb{P}(|a - Y_n| > \varepsilon) + \limsup_n \mathbb{P}(|X_n| > M)$$

The second term in RHS equals 0 by (*) for every $\varepsilon > 0$.

Now choose $M = 1/\sqrt{\varepsilon}$ and let $\varepsilon \to 0$. By (**), the third term in RHS $\to 0$, and, ~~by continuity~~, $\varepsilon M \to 0$, so by continuity

$$\limsup_n \mathbb{P}(Y_n X_n \leq x) \leq \mathbb{P}(aX \leq x).$$

The converse inequality is similarly proved (for lim inf).

QED

④

Write the ratio as

$$R_n := \frac{\frac{1}{\sqrt{n}} \sum_1^n X_k}{\left(\frac{1}{n}\sum_1^n X_k^2\right)^{1/2}}$$

With $\sigma^2 = \mathbb{E} X_k^2$, the Strong LLN gives

$$(*) \qquad \frac{1}{n}\sum_1^n X_k^2 \to \mathbb{E} X_k^2 = \sigma^2, \qquad \text{a.s.}$$

while CLT gives

$$\frac{1}{\sqrt{n}} \sum_1^n X_k \longrightarrow N \qquad \text{in distribution,}$$

where N is $N(0, \sigma^2)$.

Now, $(*)$ clearly yields

$$\frac{1}{\left(\frac{1}{n}\sum_1^n X_k^2\right)^{1/2}} \to \frac{1}{\sigma} \qquad \text{a.s.}$$

By the result in Problem 3, we get

$$R_n \to \frac{N}{\sigma} \qquad \text{in distribution.}$$

Note that $\frac{N}{\sigma}$ is $N(0,1)$. QED.