Homework 6

Probability Theory (MATH 235B, Winter 2008)

1. Let X be a random variable with mean zero and $|X| \le 1$. Prove that

$$\mathbb{E}e^{\lambda X} \le e^{\lambda^2/2}, \qquad \lambda > 0.$$

2. Let (X_k) be i.i.d. random variables with mean zero and finite positive variance. Consider $S_n = X_1 + \cdots X_n$. Prove that

$$\limsup \frac{S_n}{\sqrt{n}} = \infty \quad \text{a.s.}$$

(Hint: use the Central Limit Theorem and Kolmogorov's zero-one law; consider the event that $\limsup S_n/\sqrt{n} > M$*).*

3. Let (X_n) and (Y_n) be sequences of random variables, and a be a constant. Prove that if $X_n \to X$ in distribution and $Y_n \to a$ in probability then then $X_n Y_n \to aX$ in distribution.

4. Prove the following self-normalized version of the Central Limit Theorem. Let (X_k) be i.i.d. random variables with mean zero and finite variance. Then the self-normalized sum

$$\frac{\sum_{k=1}^n X_k}{\left(\sum_{k=1}^n X_k^2\right)^{1/2}}$$

converges in distribution, as $n \to \infty$, to the standard normal random variable. *(Hint: use Problem 3).*

5. Construct an example where Lindeberg's condition holds but Lyuapunov's does not.