1. Let \( \varphi = \varphi_x \) be the ch.f. of \( X \) (and \( Y \)). Then

\[
\varphi_{X+Y}(t) = \varphi_{X+Y}(t/\sqrt{2}) = \varphi_X(t/\sqrt{2}) \varphi_Y(t/\sqrt{2}) = \left( \varphi(t/\sqrt{2}) \right)^2 \quad (\star)
\]

**Sufficiency**  If \( X \) and \( Y \) are \( \mathcal{N}(0,1) \) then \( \varphi(t) = e^{-t^2/2} \)

Then (\( \star \)) and implies that

\[
\varphi(t) = \left( \varphi(t/\sqrt{2}) \right)^2 = e^{-t^2/4}
\]

Hence \( X+Y/\sqrt{2} \) is \( \mathcal{N}(0,1) \)

**Necessity** (\( \star \)) implies

\[
\varphi(t) = \left( \varphi(t/\sqrt{N}) \right)^N \quad \text{for all } t.
\]

Iterating this identity, we obtain

\[
\varphi(t) = \left( \varphi(t/\sqrt{N}) \right)^N \quad \text{for } N = 2^k, \quad k = 1, 2, \ldots
\]

Note that by Corollary 22.15 (second-order approximation),

\[
\left( \frac{t^2}{2N} \right)^2 < 1, \quad \varphi(t/\sqrt{N}) = 1 - \frac{t^2}{2N} + o\left( \frac{t^2}{N} \right), \quad N \to \infty
\]

\[
\Rightarrow \left( \varphi(t/\sqrt{N}) \right)^N = \left( 1 - \frac{t^2}{2N} + o\left( \frac{t^2}{N} \right) \right)^N \to e^{-t^2/2} \quad \text{as } N \to \infty
\]

\[
\Rightarrow \varphi(t) = e^{-t^2/2}.
\]

\[
\Rightarrow X \text{ is } \mathcal{N}(0,1), \quad \text{Q.E.D.}
\]
Let $X$ be a standard normal vector in $\mathbb{R}^n$, and consider

$$Z := \frac{X}{1^{\top} X}.$$

$Z$ is a random vector in $\mathbb{R}^n$, and its values are on $S^{n-1}$.

It remains to show that $Z$ is rotationally invariant.

We can realize $Z$ as $Z = f(X)$ where

$$f: \mathbb{R}^n \to S^{n-1}, \quad f(x) = \frac{x}{1^{\top} x}.$$

Then, for every Borel set $B \subset S^{n-1}$, we have and every $U \in O(n)$, we have

$$P(Z \in UB) = P(f(X) \in UB) = P(X \in f^{-1}(UB)).$$

Lemma

$$f^{-1}(UB) = UF^{-1}(B).$$

Proof: It is enough to show that $f(x) \in UB$ iff $f(U^{-1}x) \in B$

But

$$\frac{U^{-1}x}{1^{\top} U^{-1}x} = \frac{U^{-1}x}{1^{\top} x} \in B \iff \frac{x}{1^{\top} x} \in UB \implies UB \implies \text{QED}.$$

Hence

$$P(Z \in UB) = P(X \in UF^{-1}(B)) = P(X \in f^{-1}(B)) \quad \text{(by rotation invariance of $X$)} = P(f(X) \in B) = P(Z \in B).$$

Therefore, $Z$ is rotationally invariant.

Remark. $f(B)$ is the cone generated by $B$. 

-2-
By the argument of Problem 2, we can realize the random vector $X^{(n)}$ as
$$X^{(n)} = \frac{Y^{(n)}}{|Y^{(n)}|}$$
where $Y^{(n)}$ is the standard normal random vector in $\mathbb{R}^n$.

By the Strong Law of Large Numbers,
$$\frac{1}{n} |Y^{(n)}| \rightarrow 1 \quad a.s.$$
(because $|Y^{(n)}|^2 = \sum_{k=1}^{n} (Y_k^{(n)})^2$ is a sum of $n$ independent r.v.'s with mean 1, finite variance $\Rightarrow \frac{1}{n} |Y^{(n)}|^2 \rightarrow 1 \quad a.s.$)

Hence
$$\sqrt{n} X_1^{(n)} = \frac{Y_1^{(n)}}{|Y^{(n)}|/\sqrt{n}} \rightarrow N \quad \text{in distribution}$$
(see e.g. Problem 3 in KW6, or let $Y_1^{(n)}$ be the same r.v. for all $n$).

4
A centered normal vector $Y$ has the form $Y = AX$, where $X$ is a standard normal random vector, and $A$ is an invertible linear map.

If $B$ is any invertible linear map then
$$BY = BAX,$$
and $BA$ is invertible. Hence $BY$ is a centered normal random vector.

5
False Counterexample (by Don Baranauskas):
$$X = \text{standard normal}, \quad Y = X \cdot Z$$
where $Z$ is $\pm 1$ symmetric and independent of $X$. Then
$$E(X \cdot Z)(Y - EY) = E(XY) = E(X^2 \cdot Z) = E(X^2) \cdot EZ = 0$$
but clearly $X$ and $Y$ are not independent since $|X| = |Y|$. QED.