HOMEWORK 7 SOLUTIONS

①

let $\varphi = \varphi_X$ be the ch. f. of X (and Y). Then

$$\varphi_{(X+Y)/\sqrt{2}}(t) = \varphi_{X+Y}(t/\sqrt{2}) = \varphi_X(t/\sqrt{2})\,\varphi_Y(t/\sqrt{2}) = \left(\varphi(t/\sqrt{2})\right)^2 \qquad (*)$$

Sufficiency. If X and Y are $N(0,1)$ then $\varphi(t) = e^{-t^2/2}$

~~Then (*) and + imp Then (*) follows.~~

Then (*) implies that

$$\varphi_{\frac{X+Y}{\sqrt{2}}}(t) = \left(\varphi(t/\sqrt{2})\right)^2 = e^{-t^2/2}$$

Hence $\frac{X+Y}{\sqrt{2}}$ is $N(0,1)$

Necessity. (*) implies

$$\varphi(t) = \left(\varphi(t/\sqrt{2})\right)^2 \quad \text{for all } t.$$

Iterating this identity, we obtain

$$\varphi(t) = \left(\varphi(t/\sqrt{N})\right)^N \quad \text{for } N = 2^k, \quad k = 1, 2, \ldots$$

~~Note that~~ By Corollary 22.15 (second-order approximation),

$$\varphi(t/\sqrt{N}) = 1 - \frac{t^2}{2N} + o\left(\frac{t^2}{N}\right), \quad N \to \infty$$

$$\Rightarrow \left(\varphi(t/\sqrt{N})\right)^N = \left(1 - \frac{t^2}{2N} + o\left(\frac{t^2}{N}\right)\right)^N \to e^{-t^2/2} \quad \text{as } N \to \infty$$

$$\Rightarrow \varphi(t) = e^{-t^2/2}.$$

$\Rightarrow$ X is $N(0,1)$. QED

Let X be a standard normal vector in $\mathbb{R}^n$, and consider

$$Z := \frac{X}{|X|}$$

Z is a random vector in $\mathbb{R}^n$, and its values are on S^{n-1}.

It remains to show that Z is rotationally invariant.

We can realize Z as $Z = f(X)$ where

$$f : \mathbb{R}^n \to S^{n-1}, \qquad f(x) = \frac{x}{|x|}.$$

Then, for every Borel set $B \subset S^{n-1}$, ~~we have~~ and every $U \in O(n)$, we have

$$\mathbb{P}(Z \in UB) = \mathbb{P}(f(X) \in UB)$$
$$= \mathbb{P}(X \in f^{-1}(UB)).$$

Lemma $f^{-1}(UB) = U f^{-1}(B)$.

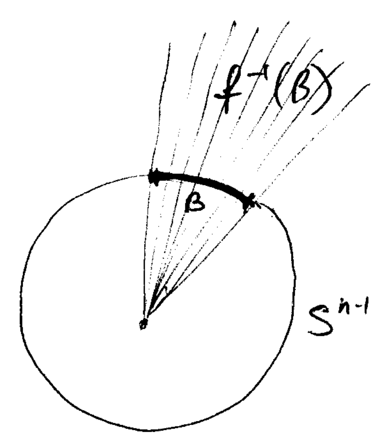
Proof: It is enough to show that $f(x) \in UB$ iff $f(U^{-1}x) \in B$

But $\dfrac{U^{-1}x}{|U^{-1}x|} = \dfrac{U^{-1}x}{|x|} \in B$ iff $\dfrac{x}{|x|} \in UB$ $\Rightarrow$ QED.

Hence

$$\mathbb{P}(Z \in UB) = \mathbb{P}(X \in U f^{-1}(B))$$
$$= \mathbb{P}(X \in f^{-1}(B)) \qquad \text{(by rotation invariance of } X)$$
$$= \mathbb{P}(f(X) \in B)$$
$$= \mathbb{P}(Z \in B).$$

Therefore, Z is rotation invariant.



Remark: $f^{-1}(B)$ is the cone generated by B.

③

By the ~~result~~ argument of Problem 2, we can realize the random vector $X^{(n)}$

as $$X^{(n)} = \frac{Y^{(n)}}{|Y^{(n)}|}$$ where $Y^{(n)}$ is the standard normal random vector in $\mathbb{R}^n$.

By the Strong Law of Large Numbers,

$$|Y^{(n)}|/\sqrt{n} \to 1 \quad \text{a.s.}$$

(because $|Y^{(n)}|^2 = \sum_{k=1}^{n} (Y_k^{(n)})^2$ is a sum of n independent r.v.'s with mean 1 ~~and finite variance~~ $\Rightarrow \frac{1}{n}|Y^{(n)}|^2 \to 1$ a.s.)

Hence $$\sqrt{n}\, X_1^{(n)} = \frac{Y_1^{(n)}}{|Y^{(n)}|/\sqrt{n}} \longrightarrow N \text{ in distribution}$$

(see e.g. Problem 3 in HW 6, or let $Y_1^{(n)}$ be the same r.v. for all n).

④

A centered normal vector Y has the form $Y = AX$, where X is a standard normal random vector, and A is an invertible linear map.

If B is any invertible linear map then

$$BY = BAX,$$

and BA is invertible. Hence BY is a centered normal random vector.

⑤

False. Counterexample (by Don Barkauskas):

$$X = \text{standard normal}, \quad Y = X \cdot Z$$

where Z is ± 1 symmetric and independent of X. Then

$$\mathbb{E}(X - \mathbb{E}X)(Y - \mathbb{E}Y) = \mathbb{E}(XY) = \mathbb{E}(X^2 \cdot Z) = \mathbb{E}(X^2)\, \mathbb{E}Z = 0$$

but clearly X and Y are not independent since $|X| = |Y|$. QED.