Homework 7

Probability Theory (MATH 235B, Winter 2008)

1. Suppose that X ad Y are independent, identically distributed random variables with mean 0 and variance 1. Prove that

$$\frac{X+Y}{\sqrt{2}} = X \qquad \text{in distribution}$$

is and only if X and Y are standard normal. *(Hint: For the necessity part, express the characteristic function of* $(X+Y)/\sqrt{2}$ *in terms of the characteristic function on* X*. Iterate this identity many times.)*

2. Prove that there exists a rotation invariant probability measure on the unit Euclidean sphere S^{n-1}. *(Complete the proof in the notes. Do not use limit theorems.)*

3. (Maxwell-Borel-Poincare Projective Central Limit Theorem) Let $X^{(n)}$ be a random vector uniformly distributed on the unit Euclidean sphere S^{n-1}. Prove that the coordinates of $X^{(n)}$ are asymptotically normal. Say, prove for the first coordinate:

$$\sqrt{n} X_1^{(n)} \to N \qquad \text{in distribution}$$

where N is the standard normal random variable. *(Hint: you may use Problem 2.)*

4. Show that an invertible linear transformation of a centered normal random vector (not necesarily standard) is a centered normal random vector.

5. Prove or disprove: normal random variables are independent if and only if they are uncorrelated. *(We say that random variables* X *and* Y *are uncorrelated if the covariance* $\mathbb{E}(X-\mathbb{E}X)(Y-\mathbb{E}Y)=0$*.)*