HOMEWORK 8 Solutions

(1)

Let $\Omega = \{a, b, c\}$; $\mathcal{F}_1 = \sigma(\{a\})$, $\mathcal{F}_2 = \sigma(\{b\})$.

Then $E(X|\mathcal{F}_1)$ is averaging over $\{b, c\}$, ~~while leaving~~
$E(X|\mathcal{F}_2)$ is averaging over $\{a, c\}$.

Consider a uniform measure on Ω,
and let X be the random variable defined as
$$X(a) = 1, \quad X(b) = 0, \quad X(c) = 0.$$

Then $Y := E(X|\mathcal{F}_1)$ has the following values:
$$Y(a) = 1, \quad Y(b) = 0, \quad Y(c) = 0$$

and $Z := E(Y|\mathcal{F}_2)$ has the following values:
$$Z(a) = \tfrac{1}{2}, \quad Z(b) = 0, \quad Z(c) = \tfrac{1}{2}.$$

On the other hand,

$Y' := E(X|\mathcal{F}_2)$ is as follows:
$$Y'(a) = \tfrac{1}{2}, \quad Y'(b) = 0, \quad Y'(c) = \tfrac{1}{2},$$

$Z' := E(X|\mathcal{F}_1)$ is as follows:
$$Z'(a) = \tfrac{1}{2}, \quad Z'(b) = \tfrac{1}{4}, \quad Z'(c) = \tfrac{1}{4}.$$

So, $Z \neq Z'$. QED.

2

(a) Let X, Y be independent. We claim that

$$E(X \mid \sigma(Y)) = EX$$

To verify this, we check the definition of the conditional expectation.

(i) Clearly, EX is measurable w.r.to any ~~σ-field~~, σ-algebra, in particular $\sigma(Y)$

(ii) Let $A \in \sigma(Y)$. Then

$$\int_A X \, dP = \mathbb{E}(X \mathbb{1}_A) = \mathbb{E}X \cdot \mathbb{E}\mathbb{1}_A \qquad \text{(by independence)}$$

$$= \mathbb{E}X \cdot \mathbb{P}(A) = \int_A (\mathbb{E}X) \, dP.$$

QED.

(b) If (X, Y) takes the values $(0,0)$, $(-1,1)$, and $(1,1)$ with probability $1/3$ each, then X and Y are clearly dependent but $E(X \mid Y) = E(X) = 0$.

~~3~~

By identical distribution,

$$\mathbb{E}(X \mid X+Y) = \mathbb{E}(Y \mid X+Y).$$

By adding these two, and by linearity,

$$\mathbb{E}(X \mid X+Y) + \mathbb{E}(Y \mid X+Y) = \mathbb{E}(X+Y \mid X+Y) = X+Y.$$

The conclusion follows (without even using independence!)

See a corrected solution (next page) →

③

We first claim that

$$E(X \mid X+Y) = E(Y \mid X+Y) \qquad (*)$$

By the definition of conditional expectation, (*) would follow if we show that

$$\int_{\{X+Y\in B\}} X \, dP = \int_{\{X+Y\in B\}} Y \, dP, \qquad \text{for every Borel } B \subset \mathbb{R},$$

as the sets $\{X+Y \in B\}$ form the σ-algebra $\sigma(X+Y)$.
Denote the distribution function of X (and of Y) by $F(x)$. Then

$$\begin{aligned}
\int_{\{X+Y\in B\}} X \, dP &= \iint_{\{x+y\in B\}} x \, dF(x)\, dF(y) && \text{(by independence)} \\
&= \iint_{\{x+y\in B\}} y \, dF(x)\, dF(y) && \text{(by change of variables } x \mapsto y \text{ and } y \mapsto x\text{)} \\
&= \int_{\{X+Y\in B\}} Y \, dP.
\end{aligned}$$

~~By a~~ This proves (*)

By adding the two terms of (*) and linearity,

$$E(X \mid X+Y) + E(Y \mid X+Y) = E(X+Y \mid X+Y) = X+Y.$$

The conclusion follows.

The assumptions and the ~~pro~~ definition of the conditional expectation imply that X and Y are $\mathcal{F}$-measurable, and

$$\int_A X \, dP = \int_A Y \, dP \qquad \text{for every } A \in \mathcal{F}.$$

$$\Rightarrow \int_A (X-Y)\, dP = 0 \qquad \forall A \in \mathcal{F}.$$

Let $\varepsilon > 0$, and consider $A_\varepsilon := \{ X - Y \geq \varepsilon \}$. Clearly $X-Y$ ~~is~~ $\mathcal{F}$-measurable, so $A_\varepsilon \in \mathcal{F}$. It follows that

$$0 = \int_{A_\varepsilon} (X-Y)\, dP \geq \varepsilon \cdot P(A_\varepsilon).$$

Hence $P(A_\varepsilon) = 0 \quad \forall \varepsilon > 0$. Therefore, $P(X-Y \geq 0) = \lim_\varepsilon P(A_\varepsilon) = 0$
(by continuity of prob. measures).

Similarly, one can show that $P(X-Y \leq 0) = 0$. Thus $P(X \neq Y) = 0$.