Homework 8

Probability Theory (MATH 235B, Winter 2008)

1. Construct an example on a three-point space Ω in which

$$\mathbb{E}(\mathbb{E}(X|\mathcal{F}_1)|\mathcal{F}_2) \neq \mathbb{E}(\mathbb{E}(X|\mathcal{F}_2)|\mathcal{F}_1)$$

for some σ-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$.

2. Show that if X and Y are independent random variables then $\mathbb{E}(X|Y) = \mathbb{E}(X)$. Show that the converse statement does not hold.

3. Let X and Y be independent, identically distributed random variables with finite mean. Show that

$$\mathbb{E}(X|X+Y) = \frac{X+Y}{2}.$$

4. Define the *conditional variance* as

$$\mathrm{var}(X|\mathcal{F}) = \mathbb{E}(X^2|\mathcal{F}) - \mathbb{E}(X|\mathcal{F})^2.$$

Show that

$$\mathrm{var}(X) = \mathbb{E}(\mathrm{var}(X|\mathcal{F})) + \mathrm{var}(\mathbb{E}(X|\mathcal{F})).$$

5. Suppose that random variables X and Y satisfy

$$\mathbb{E}(X|\mathcal{F}) = Y, \qquad \mathbb{E}(Y|\mathcal{F}) = X$$

for some σ-algebra $\mathcal{F}$. Show that $X = Y$ a.s.