Midterm Solutions / key

(1)

let $X_n \to X$ in distribution.

let $\varepsilon > 0$. Choose $M > 0$ s.t $\mathbb{P}(|X| > M) < \varepsilon$.

By the convergence in distribution,

$$\mathbb{P}(|X_n| > M) \to \mathbb{P}(|X| > M) \quad \text{as } n \to \infty.$$

Hence $\exists$ n_0 s.t.

$$\mathbb{P}(|X_n| > M) < \varepsilon \quad \text{for } n \geq n_0.$$

Then $(X_n)_{n \geq n_0}$ is tight, and $\underset{\substack{\uparrow \\ (\text{Exercise})}}{\underline{\text{consequently}}}$ $(X_n)_{n \geq 1}^{\infty}$ is tight.

(2)

$$\varphi_{X_\lambda}(t) = \exp(\lambda(e^{it} - 1)).$$

Hence for $S_\lambda := \dfrac{X_\lambda - \lambda}{\sqrt{\lambda}}$, we have

$$\varphi_{S_\lambda}(t) = \varphi_{X_\lambda/\sqrt{\lambda}}(t)\, e^{-it\sqrt{\lambda}} = \varphi_{X_\lambda}(t/\sqrt{\lambda})\, e^{-it\sqrt{\lambda}}$$

$$= \exp\left[\lambda(e^{it/\sqrt{\lambda}} - 1) - it\sqrt{\lambda}\right].$$

We use $e^{it/\sqrt{\lambda}} = 1 - \dfrac{it}{\sqrt{\lambda}} - \dfrac{t^2}{2\lambda} + O(1/\lambda^{3/2})$ as $\lambda \to \infty$.

Thus $\lambda(e^{it/\sqrt{\lambda}} - 1) - it\sqrt{\lambda} = -\dfrac{t^2}{2} + O(1/\lambda^{1/2})$ as $\lambda \to \infty$

Hence $\varphi_{S_\lambda}(t) \to e^{-t^2/2}$ as $\lambda \to \infty$.

By the Continuity Theorem, $S_\lambda \to N$.

(3)

Let F_n and G denote the distribution functions of Y_n and X, respectively, let $Z_n = X_n - Y_n$, and let x be the continuity point of G. Then

$$F_n(x) = \mathbb{P}(Y_n \le x) = \mathbb{P}(X_n \le x + Z_n)$$

$$= \mathbb{P}(X_n \le x + Z_n,\ Z_n < \varepsilon) + \mathbb{P}(X_n \le x + Z_n,\ Z_n \ge \varepsilon)$$

$$\le \mathbb{P}(X_n \le x + \varepsilon) + \mathbb{P}(Z_n \ge \varepsilon).$$

This yields
$$\limsup_n F_n(x) \le G(x+\varepsilon).$$

A similar argument shows that
$$\limsup_n F_n(x) \ge G(x-\varepsilon).$$

Since ε is arbitrary and x is a point of continuity of G, we have shown that
$$\lim F_n(x) = G(x),$$

i.e. $Y_n \to X$ in distribution.

(4)

Let $X_k = f(x_k)$, $Y_k = g(x_k)$, regarded as random variables on $[0,1]^{\mathbb{N}}$. Then (X_k) are i.i.d. r.v's, (Y_k) are i.i.d. r.v.'s.

The problem reduces to proving that

$$\mathbb{E}\left(\frac{X_1 + \dots + X_n}{Y_1 + \dots + Y_n}\right) \to \frac{\mathbb{E}X_1}{\mathbb{E}Y_1}, \quad \text{as } n \to \infty. \qquad (**)$$

By Strong L.L.N.,
$$\frac{1}{n}(X_1 + \dots + X_n) \to \mathbb{E}X_1 \quad \text{a.s.},$$
$$\frac{1}{n}(Y_1 + \dots + Y_n) \to \mathbb{E}Y_1 \quad \text{a.s.}$$

Hence
$$\frac{X_1+\dots+X_n}{Y_1+\dots+Y_n} \to \frac{\mathbb{E}X_1}{\mathbb{E}Y_1} \quad \text{a.s.} \qquad (*)$$

The assumption that $0 \le X_1 \le C Y_1$ implies that LHS in $(*)$ is bounded by C. Hence the Dominated Convergence Theorem implies ~~that~~ $(**)$, and completes the proof.

⑤

By CLT,
$$\mathbb{P}\left(\frac{S_n}{\sigma\sqrt{n}} \le x\right) \to \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt \qquad \text{as } n\to\infty.$$

Hence
$$\mathbb{P}\left(\frac{S_n}{\sqrt{n}} \le 1\right) = \mathbb{P}\left(\frac{S_n}{\sigma\sqrt{n}} \le \frac{1}{\sigma}\right) \to \Phi\left(\frac{1}{\sigma}\right).$$

So, by the assumption,
$$\Phi\left(\frac{1}{\sigma}\right) = \frac{2}{3}.$$

From a table of standard normal distribution, we find
$$1/\sigma \approx 0.43, \quad \text{so} \quad \boxed{\sigma^2 \approx 5.41}$$

⑥

The formula for the total probability yields
$$\mathbb{P}\left(\frac{S_{\nu_n}}{\sqrt{\nu_n}} \le x\right) = \sum_{k=1}^{\infty} \mathbb{P}(\nu_n = k)\cdot \mathbb{P}\left(\frac{S_k}{\sqrt{k}} \le x\right).$$

By CLT, $\quad \mathbb{P}\left(\frac{S_k}{\sqrt{k}} \le x\right) \to \mathbb{P}(N \le x) \quad$ as $k\to\infty$.

This implies that for arbitrary $\varepsilon > 0$ there exists k_ε such that
$$\left|\mathbb{P}\left(\frac{S_k}{\sqrt{k}} \le x\right) - \mathbb{P}(N \le x)\right| < \frac{\varepsilon}{2} \quad \text{for all } k \ge k_\varepsilon.$$

Furthermore, $\nu_n \to \infty$ in probability, which means that

$$\forall M, \quad \mathbb{P}(\nu_n < M) \to 0 \quad \text{as } n \to \infty.$$

So there exists n_ε such that

$$\mathbb{P}(\nu_n < k_\varepsilon) < \frac{\varepsilon}{2} \quad \text{for all } n \geq n_\varepsilon.$$

Therefore, for $n \geq n_\varepsilon$, we have

$$\left| \mathbb{P}\left(\frac{S_{\nu_n}}{\sqrt{\nu_n}} \leq x \right) - \mathbb{P}(N \leq x) \right| = \left| \sum_{k=1}^{\infty} \mathbb{P}(\nu_n = k) \cdot \mathbb{P}\left(\frac{S_k}{\sqrt{k}} \leq x \right) - \mathbb{P}(N \leq x) \right|$$

$$\leq \underbrace{\mathbb{P}(\nu_n < k_\varepsilon)}_{< \frac{\varepsilon}{2}} + \sum_{k=k_\varepsilon}^{\infty} \mathbb{P}(\nu_n = k) \underbrace{\left| \mathbb{P}\left(\frac{S_k}{\sqrt{k}} \leq x \right) - \mathbb{P}(N \leq x) \right|}_{< \varepsilon/2}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \sum_{k=k_\varepsilon}^{\infty} \mathbb{P}(\nu_n = k) \leq \varepsilon.$$

Q.E.D.