Midterm 1. Probability Theory (MATH 235B, Winter 2008)

Independent work. Please only use: (1) class or HW material; (2) tables or computer generated values of the standard probability distributions.

1. Prove that a sequence of random variables that converges in distribution is tight.

2. Recall that a Poisson random variable X_λ with parameter $\lambda > 0$ has mean λ and standard deviation $\sqrt{\lambda}$. Prove that

$$\frac{X_\lambda - \lambda}{\sqrt{\lambda}} \to N \quad \text{in distribution as } \lambda \to \infty,$$

where N is a standard normal random variable.

3. Let (X_n) and (Y_n) be sequences of random variables such that $X_n - Y_n \to 0$ in probability. Prove that if $X_n \to X$ in distribution then $Y_n \to X$ in distribution.

4. Let $f(x)$ and $g(x)$ be real valued and continuous functions on the interval $[0,1]$, which satisfy $0 \le f(x) \le Cg(x)$ for some constant $C > 0$. Prove that

$$\int_0^1 \cdots \int_0^1 \frac{f(x_1) + \cdots + f(x_n)}{g(x_1) + \cdots + g(x_n)} \, dx_1 \ldots dx_n \to \frac{\int_0^1 f(x)\,dx}{\int_0^1 g(x)\,dx} \qquad \text{as } n \to \infty.$$

(Hint: $X_k = f(x_k)$, $Y_k = g(x_k)$ are r.v.'s on the probability space $[0,1]^n$.)

5. Let (X_n) be a sequence of independent and identically distributed random variables with zero mean and finite but unknown variance σ^2. Suppose for the sum $S_n = X_1 + \cdots + X_n$ we know that

$$\mathbb{P}\Big(\frac{S_n}{\sqrt{n}} > 1\Big) \to \frac{1}{3} \quad \text{as } n \to \infty.$$

Determine the unknown variance σ^2.

6. Bonus problem. Full credit only. Consider a sequence of independent and identically distributed random variables (X_n) with zero mean and unit variance. Let (ν_k) be another sequence of integer-valued random variables such that ν_k does not depend on (X_n) for each k. Suppose that $\nu_k \to \infty$ in probability (define yourself what this means). Prove that the sums $S_n = X_1 + \cdots + X_n$ satisfy

$$\frac{S_{\nu_n}}{\sqrt{\nu_n}} \to N \quad \text{in distribution as } n \to \infty,$$

where N is a standard normal random variable.