Midterm Exam 2

Math 525, Prof. Roman Vershynin
Winter 2009

Name: ______________________________

Read the following information before starting the exam:

- No textbooks or note sheets are allowed on this exam. Calculators are allowed. Also allowed is one hand-written index card (3 by 5).
- The table of the distribution function $\Phi(x)$ of a standard normal random variable is included on the last page.
- Show all work, clearly and in order, if you want to get full credit. Points may be taken off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.

1. (*5 points*) It is natural to assume that the lifetime of a typical fuse is exponentially distributed. Tests show that about half of all fuses do not live longer than $2 \ln 2 \approx 1.39$ years. Compute the expected lifetime of a fuse.

2. (*5 points*) Suppose requests to a web server follow a Poisson process. During some 5 minute period, one request came. Find the probability that this request came during the first 3 minutes of the period. *Justify your answer.*

3. (*10 points*) A rat is trapped in a maze. Initially it has to choose one of the two directions. If it goes to the right, then it will wander around in the maze for 3 minutes and will then return to the its initial position. If it goes to the left, then with probability $\frac{1}{3}$ it will depart the maze after two minutes of traveling, and with probability $\frac{2}{3}$ it will return to its initial position after five minutes of traveling. Assuming that the rat is at all times equally likely to go to the left or to the right, what is the expected time that it will be trapped in the maze?

4. (*10 points*) The lifetime of Aaron's dog and cat are independent exponential random variables with respective rates λ_d and λ_c. Find the expected additional lifetime of the pet after the other pet dies. *Hint: condition on which pet dies first.*

5. (*8 points*) Physicists P. and T. Ehrenfest proposed the following model to describe the movements of molecules. Suppose N molecules are distributed among two urns. At each time point, one of the molecules is chosen at random, removed from its urn, and placed in the other one. One observes the number of molecules in the first urn at each time point.

(a) For $N = 2$ molecules, construct a Markov chain for this model (i.e. specify the transition probability matrix).

(b) Compute the limiting probabilities for this Markov chain.

6. (*7 points*) For a Markov chain with the following transition probability matrix and states $\{0, 1, 2, 3\}$, specify the classes, and determine whether they are transient or recurrent:

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

7. (*7 points*) Let (X, Y) be a point picked according to the uniform distribution in the quarter-circle $\{(x, y) \mid x \geq 0,\ y \geq 0,\ x^2 + y^2 \leq 1\}$. Find $E[X|Y]$.

Bonus Question *(8 extra points, no partial credit).*

Consider independent random variables $X_1, X_2, \ldots, X_n$ uniformly distributed on $[0, 1]$, and let $M = \max(X_1, \ldots, X_n)$. Compute $E[X_1|M]$.