Exam 1. Due Friday, October 9

Functional Analysis (602, Real Analysis II), Fall 2009

1. Prove that the following inequality holds for every two vectors x, y in a normed space X:

$$\|x\| \leq \max\big(\|x+y\|, \|x-y\|\big).$$

2. Proved that the spaces ℓ_p, $1 \leq p < \infty$ are separable, while ℓ_∞ is not.

3. Let f be a linear functional on a normed space X. Prove that f continuous if and only if the sets $\{x : \ f(x) < a\}$ and $\{x : \ f(x) > a\}$ are open for every $a \in \mathbb{R}$.

4. Prove that the space of all polynomials is dense in $L_p[0,1]$, $1 \leq p < \infty$.

5. Determine and prove whether the following normed spaces are complete or incomplete:

(i) The space of all polynomials with the norm

$$\|a_0 + a_1 t + \cdots + a_n t^n\| := |a_0| + \cdots + |a_n|.$$

(ii) The space of continuous functions on $[0,1]$ with the norm

$$\|f\| := \int_0^1 |f(t)|\, dt.$$

6. (i) Consider the subspace Y of constant functions in $C[a,b]$. Show that the norm in the quotient space $C[a,b]/Y$ can be computed as follows:

$$\|[f]\| = \frac{1}{2}\Big(\max_{t\in[a,b]} f(t) - \min_{t\in[a,b]} f(t)\Big), \qquad f \in C[a,b].$$

(ii) Show that the norm in the quotient space ℓ_∞/c_0 can be computed as

$$\|[a]\| = \limsup |a_n|, \qquad a = (a_1, a_2, \ldots) \in \ell_\infty.$$

7. Let $A \subseteq [-\pi, \pi]$ be a Lebesgue measurable set. Prove that

$$\lim_{n\to\infty} \int_A \sin(nt)\, dt = \lim_{n\to\infty} \int_A \cos(nt)\, dt = 0.$$

(Hint: espress these integrals as Fourier coefficients of some function.)