Exam 2. Due Friday, November 6

Functional Analysis (602, Real Analysis II), Fall 2009

1. (Compactness) Let A be a bounded subset of a Banach space X.

(a) Prove that A is a precompact if and only if, for every $\varepsilon > 0$ there exists a finite dimensional subspace $Y \subseteq X$ which forms an ε-net of A. *(Necessity was proved in class; please prove sufficiency).*

(b) Suppose that there exists an *approximate identity* in X, which is a sequence of operators of finite rank $T_n \in L(X, X)$ which converges pointwise to the identity: $T_n x \to x$ for every $x \in X$. Prove that A is precompact if and only if T_n converges to the identity uniformly on A. *(Hint: necessity is a direct consequence of one of the theorems proved in class; sufficiency may follow from part (a)).*

(c) For $X = c_0$, prove that A is precompact if and only if there exists an vector $x \in c_0$ which dominates all vectors in A pointwise: $|a_i| \leq |x_i|$ for all $a \in A$ and $i = 1, 2, \ldots$

(d) For $X = \ell_p$, prove that A is precompact if and only if vectors in A have uniformly decaying tails: for every $\varepsilon > 0$ there exists N such that $\sum_{i>N} |x_i|^p \leq \varepsilon$ for all $x \in A$.

2. (Banach limit) Construct a generalization of the notion of limit which is defined for all bounded (but possibly divergent) sequences. Specifically, show that to every bounded sequence of real numbers (x_n) one can assign a real number called Banach limit and denoted by $\operatorname{Lim} x_n$, which has the following properties:

(i) If x_n converges then $\operatorname{Lim} x_n = \lim x_n$;

(ii) $\liminf x_n \leq \operatorname{Lim} x_n \leq \limsup x_n$;

(iii) $\operatorname{Lim}(a x_n + b y_n) = a \operatorname{Lim} x_n + b \operatorname{Lim} y_n$ for scalars a, b;

(iv) Translations do not change the limit: $\operatorname{Lim} x_{n+1} = \operatorname{Lim} x_n$.

(Hint: Construct such an extension using Hahn-Banach theorem in the following setting: consider the Cesaro means $L_n x := \frac{1}{n}(x_1 + \cdots x_n)$, the subspace

$$Y = \{x \in \ell_\infty : \ \lim L_n x =: \operatorname{Lim} x \ \textit{exists}\},$$

and the sublinear functional $p(x) = \limsup L_n x$ on ℓ_∞).

(Over, please)

3. (Projections) Let X be a Banach space.

(i) Suppose X_1 and X_2 are closed linear subspaces of X with the following property: every $x \in X$ can be uniquely expressed as $x = x_1 + x_2$ for some $x_1 \in X_1$, $x_2 \in X_2$. Prove that X_1 and X_2 are complemented subspaces. Namely, show that the map $P_1 x := x_1$ defines a projection on X_1 and $P_2 x := x_2$ defines a projection onto X_2. *(Hint: use Closed Graph Theorem to prove boundedness.)*

(ii) Deduce that X is isomorphic to the direct sum $X_1 \oplus X_2$.

(iii) Let $P : X \to X$ be a linear operator such that $P^2 = P$. Prove that $X_1 := \operatorname{im} P$ and $X_2 := \ker P$ are closed complemented subspaces of X. Deduce that P is a projection onto X_1.