1. (Compactness) Let $A$ be a bounded subset of a Banach space $X$.

(a) Prove that $A$ is a precompact if and only if, for every $\varepsilon > 0$ there exists a finite dimensional subspace $Y \subseteq X$ which forms an $\varepsilon$-net of $A$. *(Necessity was proved in class; please prove sufficiency).*

(b) Suppose that there exists an approximate identity in $X$, which is a sequence of operators of finite rank $T_n \in L(X, X)$ which converges pointwise to the identity: $T_n x \to x$ for every $x \in X$. Prove that $A$ is precompact if and only if $T_n$ converges to the identity uniformly on $A$. *(Hint: necessity is a direct consequence of one of the theorems proved in class; sufficiency may follow from part (a)).*

(c) For $X = c_0$, prove that $A$ is precompact if and only if there exists a vector $x \in c_0$ which dominates all vectors in $A$ pointwise: $|a_i| \leq |x_i|$ for all $a \in A$ and $i = 1, 2, \ldots$

(d) For $X = \ell_p$, prove that $A$ is precompact if and only if vectors in $A$ have uniformly decaying tails: for every $\varepsilon > 0$ there exists $N$ such that $\sum_{i>N} |x_i|^p \leq \varepsilon$ for all $x \in A$.

2. (Banach limit) Construct a generalization of the notion of limit which is defined for all bounded (but possibly divergent) sequences. Specifically, show that to every bounded sequence of real numbers $(x_n)$ one can assign a real number called Banach limit and denoted by $\text{Lim } x_n$, which has the following properties:

(i) If $x_n$ converges then $\text{Lim } x_n = \lim x_n$;
(ii) $\liminf x_n \leq \text{Lim } x_n \leq \limsup x_n$;
(iii) $\text{Lim}(ax_n + by_n) = a \text{Lim } x_n + b \text{Lim } y_n$ for scalars $a, b$;
(iv) Translations do not change the limit: $\text{Lim } x_{n+1} = \text{Lim } x_n$.

*(Hint: Construct such an extension using Hahn-Banach theorem in the following setting: consider the Cesaro means $L_n x := \frac{1}{n} (x_1 + \cdots + x_n)$, the subspace $Y = \{ x \in \ell_\infty : \lim L_n x =: \text{Lim } x \text{ exists} \}$, and the sublinear functional $p(x) = \limsup L_n x$ on $\ell_\infty$).*

(Over, please)
3. **(Projections)** Let $X$ be a Banach space.

(i) Suppose $X_1$ and $X_2$ are closed linear subspaces of $X$ with the following property: every $x \in X$ can be uniquely expressed as $x = x_1 + x_2$ for some $x_1 \in X_1$, $x_2 \in X_2$. Prove that $X_1$ and $X_2$ are complemented subspaces. Namely, show that the map $P_1 x := x_1$ defines a projection on $X_1$ and $P_2 x := x_2$ defines a projection onto $X_2$. *(Hint: use Closed Graph Theorem to prove boundedness.)*

(ii) Deduce that $X$ is isomorphic to the direct sum $X_1 \oplus X_2$.

(iii) Let $P : X \to X$ be a linear operator such that $P^2 = P$. Prove that $X_1 := \text{im } P$ and $X_2 := \ker P$ are closed complemented subspaces of $X$. Deduce that $P$ is a projection onto $X_1$. 