Exam 2. Due Friday, December 11

Functional Analysis (602, Real Analysis II), Fall 2009

1. (Separability) Let X be a Banach space. Prove that if X^* is separable then X is separable, too. Is the converse true?

2. (General form of spectrum) Prove that every nonempty compact subset of the complex plane is the spectrum of some bounded linear operator. *(Hint: consider a multiplication (i.e. diagonal) operator on ℓ_2.)*

3. (Stability of completeness) Let (x_n) be a complete orthonormal system in a Hilbert space H. Suppose that a system of elements (y_n) is a perturbation of (x_n) in the sense that

$$\sum_{n=1}^{\infty} \|x_n - y_n\|^2 < 1.$$

Show that the system (y_n) is complete. *(Hint: consider a vector orthogonal to all y_n and apply Parseval's identity.)*

4. (Exponential function of an operator) Let T be a self-adjoint operator on a Hilbert space. Prove that the operator e^T is not compact. *(Hint: using the spectral mapping theorem, show that the spectrum of e^T does not contain 0.)*

5. (Compactness of self-adjoint operators) Let $T \in L(H,H)$ be a self-adjoint operator on a Hilbert space. Show that T is compact if and only if T^2 is compact. Show by example that this result may fail for operators that are not self-adjoint.

(Over, please)

6. (Monotone convergence theorem for operators) Let $T \in L(H, H)$ be a self-adjoint linear operator on a Hilbert space.

(a) Prove a version of Cauchy-Schwarz inequality

$$\langle Tx, y \rangle \leq \langle Tx, x \rangle^{1/2} \langle Ty, y \rangle^{1/2}.$$

(b) Deduce the inequality

$$\|Tx\| \leq \|T\|^{1/2} \langle Tx, x \rangle^{1/2}.$$

(c) Consider a monotone increasing and bounded sequence of self-adjoint operators $T_n \in L(H, H)$, i.e. $T_1 \leq T_2 \leq \cdots$ and $\sup_n \|T_n\| < \infty$. Show that this sequence has a pointwise limit $T \in L(H, H)$, i.e. $T_n x \to Tx$ for all $x \in H$.

(d) Prove that the convergence in the operator norm may not hold *(Hint: consider the partial sums of a Fourier series.)*