Homework 09/11

Functional Analysis (602, Real Analysis II), Fall 2009

1. Let E and F be linear spaces, and E_1 be a subspace of E. Prove that every linear operator $T_1 : E_1 \to F$ can be extended to a linear operator $T : E \to F$ (i.e. the two operators must act identically on E_1: $T_1 x = Tx$ for all $x \in E_1$).

2. (Injectivization) Consider a linear operator $T : E \to F$ acting between linear spaces E and F. The operator T may not be injective; we would like to make it into an injective operator. To this end, we consider the map $\tilde{T} : X/\ker T \to Y$ which sends every coset $[x]$ into a vector Tx, i.e. $\tilde{T}[x] = Tx$.

(i) Prove that $\tilde{T}$ is well defined, i.e. $[x_1] = [x_2]$ implies $Tx_1 = Tx_2$.

(ii) Check that $\tilde{T}$ is a linear and injective operator.

(iii) Show that $T = \tilde{T} \circ q$, where $q : X \to X/\ker T$ is the quotient map. In other words, every linear operator can be represented as a composition of a surjective and injective operator.

3. Show that the convergence in the space $C[a, b]$ is the uniform convergence on $[a, b]$.

4. Suppose a sequence (x_n) of vectors in a normed space converges to a vector x. Show that $\|x_n\| \to \|x\|$ as $n \to \infty$.

5. Consider the linear space $\hat{c}$ of double sequences $x = (x_n)_{n=-\infty}^{\infty}$ such that the limits $b_+ = \lim_{n\to\infty} x_n$ and $b_- = \lim_{n\to-\infty} x_n$ exist. Consider the subspace $\hat{c}_0$ of the sequences for which $b_+ = b_- = 0$. Compute the codimension of $\hat{c}_0$.

6. Prove that c_0 is a closed subspace of ℓ_∞.