1. Prove that $\ell_\infty$ is a Banach space.

2. Let $X$ be a Banach space and $Y$ be a closed subspace of $X$. Prove that the quotient space $X/Y$ is a Banach space.

3. (Minkowski functional) Consider a closed, convex, origin-symmetric set $K$ in $\mathbb{R}^n$ with nonempty interior. Minkowski functional of $K$ is the function defined on $\mathbb{R}^n$ by

$$\|x\|_K := \inf \{ t > 0 : x/t \in K \}.$$ 

Show that $\| \cdot \|_K$ is a norm on $\mathbb{R}^n$, and the unit ball of this normed space is $K$.

4. ($\ell_\infty$ is the limit of $\ell_p$) Show that for every $x \in \mathbb{R}^n$ one has

$$\|x\|_p \to \|x\|_\infty \quad \text{as} \quad p \to \infty.$$ 

5. ($C^k$ spaces) Let $k \in \mathbb{N}$. Consider the space $C^k[0,1]$ which consists of the real-valued functions $f$ defined on $[0,1]$ that have derivatives up to the $k$-th order, and with the norm

$$\|f\| := \max(\|f\|_\infty, \|f'\|_\infty, \|f''\|_\infty, \ldots, \|f^{(k)}\|_\infty).$$

Prove that $C^k[0,1]$ is a Banach space.

6. Let $(x_n)$ be a Cauchy sequence in a normed space $X$. Prove that:

(i) If some subsequence of $(x_n)$ converges to a vector $x \in X$ then the whole sequence $(x_n)$ converges to $x$.

(ii) There exists a subsequence $(y_k) \subseteq (x_n)$ such that

$$\|y_{k+1} - y_k\| \leq 2^{-k}, \quad k = 1, 2, \ldots$$