Homework 09/18

Functional Analysis (602, Real Analysis II), Fall 2009

1. Prove that ℓ_∞ is a Banach space.

2. Let X be a Banach space and Y be a closed subspace of X. Prove that the quotient space X/Y is a Banach space.

3. (Minkowski functional) Consider a closed, convex, origin-symmetric set K in $\mathbb{R}^n$ with nonempty interior. Minkowski functional of K is the function defined on $\mathbb{R}^n$ by

$$\|x\|_K := \inf\big\{t > 0 \ : \ x/t \in K\big\}.$$

Show that $\|\cdot\|_K$ is a norm on $\mathbb{R}^n$, and the unit ball of this normed space is K.

4. (ℓ_∞ is the limit of ℓ_p) Show that for every $x \in \mathbb{R}^n$ one has

$$\|x\|_p \to \|x\|_\infty \quad \text{as } p \to \infty.$$

5. (C^k spaces) Let $k \in \mathbb{N}$. Consider the space $C^k[0,1]$ which consists of the real-valued functions f defined on $[0,1]$ that have derivatives up to the k-th order, and with the norm

$$\|f\| := \max(\|f\|_\infty, \|f'\|_\infty, \|f''\|_\infty, \dots, \|f^{(k)}\|_\infty).$$

Prove that $C^k[0,1]$ is a Banach space.

6. Let (x_n) be a Cauchy sequence in a normed space X. Prove that:

(i) If some subsequence of (x_n) converges to a vector $x \in X$ then the whole sequence (x_n) converges to x.

(ii) There exists a subsequence $(y_k) \subseteq (x_n)$ such that

$$\|y_{k+1} - y_k\| \le 2^{-k}, \quad k = 1, 2, \dots$$