Homework 09/18

Functional Analysis (602, Real Analysis II), Fall 2009

1. Prove that $C[0,1]$ is not dense in $L_\infty[0,1]$.

2. Prove that the inner product is a continuous function on the Cartesian product $X \times X$ of a Hilbert space X.

3. (Parallelogram law) (i) Prove that, in every inner product space X, the following identity (parallelogram law) holds:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \text{for all } x, y \in X.$$

(ii) Prove that the parallelogram law characterizes inner product spaces. Namely, suppose that the parallelogram law holds. Then one can define an inner product $\langle \cdot, \cdot \rangle$ on X in such a way that $\|x\| = \langle x, x\rangle^{1/2}$ for all $x \in X$.

Hint: consider spaces over $\mathbb{R}$ only; define the inner product via the polarization identity $\langle x, y\rangle = \frac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2\big)$.

(iii) Check that the parallelogram law does not hold in spaces $C[0,1]$, $L_1[0,1]$ and c_0.

4. (Hilbert space of matrices) Consider the linear space $M_{n,m}$ of all $m \times n$ matrices with complex entries. Show that the formula

$$\langle A, B\rangle := \operatorname{trace}(A^* B)$$

defines an inner product on $M_{m,n}$. Write out the corresponding norm $\|A\|$ in terms of the matrix entries. What does Cauchy-Schwartz inequality look in this $M_{n,m}$?

5. (Projections onto closed sets) - may be difficult. Let X be a Hilbert space and $A \subseteq X$ be a closed, convex and nonempty set. The closest vector in A to a vector $x \in X$ is denoted by $P_A x$ and called the projection of x onto A. Recall that by the general form of theorem we proved in class (p. 32 in the notes), $P_A x$ exists and is unique. Prove that the map $P_A : X \to X$ is non-expansive and hence continuous, i.e.

$$\|P_A x - P_A y\| \le \|x - y\| \quad \text{for all } x, y \in X.$$