Homework 10/09

Functional Analysis (602, Real Analysis II), Fall 2009

1. (i) Prove the identity

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

by expanding the function $f(t) = t^2$ in Fourier series in $L_2[-\pi, \pi]$ and using Parseval's identity.

(ii) In a similar way, compute $\sum_{k=1}^{\infty} 1/k^4$.

2. Prove that the system of functions $\sin nt$, $n = 1, 2, \ldots$ is an orthogonal basis of $L_2[0, \pi]$.

3. Consider the linear functional on $C[0,1]$ given by

$$F(f) = \int_0^1 f(t)g(t)\, dt, \quad f \in C[0,1]$$

where g is a fixed function.

(i) Describe the functions g for which F is a bounded linear functional;
(ii) compute the norm of F in these cases;
(iii) show by example that F may not attain its norm.

4. Consider the linear functional on c_0 given by

$$f(x) = \sum_{n=1}^{\infty} x_n/2^{n-1}, \quad x = (x_1, x_2, \ldots) \in c_0.$$

(i) Show that $f \in c_0^*$ and compute the norm of f.
(ii) Show that f foes not attain its norm.

5. (Duality between subspaces and quotient spaces) The annihilator of a subset E of a normed space X is the subspace of X^* defined as

$$E^{\perp} = \{f \in X^* : \ f(x) = 0 \text{ for all } x \in E\}.$$

Let Y be a closed subspace of a normed space X. Show that, under natural identifications (what are they?) one has:

(i) $(X/Y)^* = Y^{\perp}$ (hence the dual of a quotient space is a subspace);
(ii) $Y^* = X^*/Y^{\perp}$ (hence the dual of a subspace is a quotient space).