Homework 10/09
Functional Analysis (602, Real Analysis II), Fall 2009

1. (i) Prove the identity
\[ \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \]
by expanding the function \( f(t) = t^2 \) in Fourier series in \( L_2[-\pi, \pi] \) and using Parseval’s identity.
   (ii) In a similar way, compute \( \sum_{k=1}^{\infty} \frac{1}{k^4} \).

2. Prove that the system of functions \( \sin nt, n = 1, 2, \ldots \) is an orthogonal basis of \( L_2[0, \pi] \).

3. Consider the linear functional on \( C[0, 1] \) given by
   \[ F(f) = \int_0^1 f(t)g(t) \, dt, \quad f \in C[0, 1] \]
where \( g \) is a fixed function.
   (i) Describe the functions \( g \) for which \( F \) is a bounded linear functional;
   (ii) compute the norm of \( F \) in these cases;
   (iii) show by example that \( F \) may not attain its norm.

4. Consider the linear functional on \( c_0 \) given by
   \[ f(x) = \sum_{n=1}^{\infty} \frac{x_n}{2^{n-1}}, \quad x = (x_1, x_2, \ldots) \in c_0. \]
   (i) Show that \( f \in c_0^* \) and compute the norm of \( f \).
   (ii) Show that \( f \) does not attain its norm.

5. (Duality between subspaces and quotient spaces) The annihilator of a subset \( E \) of a normed space \( X \) is the subspace of \( X^* \) defined as
   \[ E^\perp = \{ f \in X^* : f(x) = 0 \text{ for all } x \in E \}. \]
Let \( Y \) be a closed subspace of a normed space \( X \). Show that, under natural identifications (what are they?) one has:
   (i) \( (X/Y)^* = Y^\perp \) (hence the dual of a quotient space is a subspace);
   (ii) \( Y^* = X^*/Y^\perp \) (hence the dual of a subspace is a quotient space).