Homework 10/16

Functional Analysis (602, Real Analysis II), Fall 2009

1. Geometric form of Hahn-Banach Theorem. Based on the separation theorem for a set and a point (proved in Lecture 15), prove the following geometric form of Hahn-Banach theorem. Let A, B be disjoint convex subsets of a normed space X, and assume that A is open. Then there exists $f \in X^*$ and $C \in \mathbb{R}$ such that

$$f(a) < C \leq f(b) \quad \text{for all } a \in A, b \in B.$$

(Hint: consider the set $K = A - B := \{a - b : \ a \in A, b \in B\}$ *and use that* $0 \notin K$.

2. Limitations of Hahn-Banach Theorem. Here you will construct two convex disjoint sets which can not be separated. This will show that the assumption in Problem 1 that one of the sets is open can not be dropped.

Consider the linear space $\mathcal{P}$ of all polynomials in one variable and with real coefficients. Let the subset A consist of polynomials with negative leading coefficient, and let the subset B consists of polynomials with all nonnegative coefficients. Show that A and B are disjoint convex subsets of $\mathcal{P}$, and that there does **not** exist a nonzero linear functional f on $\mathcal{P}$ such that

$$f(a) \leq f(b) \quad \text{for all } a \in A, b \in B.$$

3. Integral operators. (i) Show that the integral operator T on $L_2[0,1]$,

$$(Tf)(t) = \int_0^1 k(t,s) f(s) \, ds$$

has norm $\|T\| = \|k\|_2$ (the upper bound was proved in Lecture 16).

(ii) Compute the norm of the integral operator on $C[0,1]$, assuming that the kernel $k \in C([0,1]^2)$.

4. Invertibility. Suppose $T \in L(X,X)$ and $\|T\| < 1$. Prove that $\mathrm{Id} - T$ is invertible, $(\mathrm{Id} - T)^{-1} \in L(X,X)$ and

$$(\mathrm{Id} - T)^{-1} = I + T + T^2 + T^3 + \cdots$$