Homework 10/23

Functional Analysis (602, Real Analysis II), Fall 2009

1. Compute the norms of the following linear operators:

(i) An integral operator T on $L_2[0,1]$ defined as

$$(Tf)(t) = \int_0^1 k(t,s)f(s)\,ds$$

for some kernel $k(t,s) \in L_2([0,1]^2)$.

(ii) A similar integral operator on $C[0,1]$, with some kernel $k(t,s) \in C([0,1]^2)$.

(iii) The multiplication operator on T on $C[0,1]$ defined as

$$(Tf)(t) = k(t)f(t),$$

for some multiplier $k(t) \in C[0,1]$. For which multipliers $k(t)$ is the operator T injective? For which $k(t)$ is it surjective?

2. Let X, Y be normed spaces, and let $T \in L(X,Y)$. Prove that

(i) $\|T^*\| = \|T\|$;

(ii) $(\operatorname{Im} T)^\perp = \ker T^*$;

(iii) $\ker T = (\operatorname{Im} T^*)_\perp$.

We used here the standard notation for annihilators: $A^\perp = \{f \in X^* : \ f(x) = 0 \text{ for all } x \in A\}$ for $A \subset X$ and $B_\perp = \{x \in X : \ f(x) = 0 \text{ for all } f \in B\}$ for $B \subset X^*$.

3. (i) Prove that every proper closed linear subspace of a normed space is a nowhere dense set.

(ii) Consider the linear space c_{00} which consists of sequences $(x_i)_{i=1}^n$ with finitely many nonzero coordinates. Using (i) and Baire category theorem, show that c_{00} is not a Banach space with respect to any norm.

4. Let $T : X \to Y$ be a linear operator, where X and Y are normed spaces. Suppose that X is finite dimensional; prove that T is bounded.