1. (i) Prove the following strengthening of Mazur’s lemma. Let \( x_n \) be a sequence in a Banach space \( X \) which converges weakly to \( x \in X \). Then
\[
\{x\} = \bigcap_{n \geq 1} \overline{\text{conv}(x_i)_{i \geq n}}
\]

(ii) Consider the sequence \( x_n = (1, \ldots, 1, 0, 0, \ldots) \) (with \( n \) ones and rest zeros) in \( \ell_\infty \). Use Mazur’s lemma to show that \( x_n \) does not weakly converge. Deduce that the criterion of weak convergence in spaces \( \ell_p \) \((1 < p < \infty)\) and \( c_0 \) (i.e. boundedness and pointwise convergence) does not hold for \( \ell_\infty \).

2. Prove that the uniform measures on the intervals \([x_0 - \frac{1}{n}, x_0 + \frac{1}{n}]\) weakly (i.e. weak\(^*\)) converge to the Dirac delta function \( \delta_{x_0} \).

3. Let \( K \) be a w-compact set in a Banach space \( X \), and let \( F : K \to \mathbb{R} \) be a convex and continuous function. Show that
\[
\sup_{x \in K} f(x) = \sup_{x \in \text{ext}(K)} f(x).
\]

4. Prove that:
   (i) \( \text{ext}(B_{c_0}) = \emptyset \);
   (ii) \( \text{ext}(B_{C[0,1]}) = \{-1, 1\} \);
   (iii) \( \text{ext}(B_{L_1[0,1]}) = \emptyset \);
   (iv) for \( K \) being the unit ball of the operator space \( L(\ell^2, \ell^2) \), the extremal points of \( K \) are precisely the unitary operators.