Homework 11/06

Functional Analysis (602, Real Analysis II), Fall 2009

1. (i) Prove the following strengthening of Mazur's lemma. Let x_n be a sequence in a Banach space X which converges weakly to $x \in X$. Then

$$\{x\} = \bigcap_{n \geq 1} \overline{\mathrm{conv}(x_i)_{i \geq n}}$$

(ii) Consider the sequence $x_n = (1, \ldots, 1, 0, 0, \ldots)$ (with n ones and rest zeros) in ℓ_∞. Use Mazur's lemma to show that x_n does not weakly converge. Deduce that the criterion of weak convergence in spaces ℓ_p $(1 < p < \infty)$ and c_0 (i.e. boundedness and pointwise convergence) does not hold for ℓ_∞.

2. Prove that the uniform measures on the intervals $[x_0 - \frac{1}{n}, x_0 + \frac{1}{n}]$ weakly (i.e. weak*) converge to the Dirac delta function δ_{x_0}.

3. Let K be a w-compact set in a Banach space X, and let $F : K \to \mathbb{R}$ be a convex and continuous function. Show that

$$\sup_{x \in K} f(x) = \sup_{x \in \mathrm{ext}(K)} f(x).$$

4. Prove that:

(i) $\mathrm{ext}(B_{c_0}) = \emptyset$;

(ii) $\mathrm{ext}(B_{C[0,1]}) = \{-1, 1\}$;

(iii) $\mathrm{ext}(B_{L_1[0,1]}) = \emptyset$;

(iv) for K being the unit ball of the operator space $L(\ell_2^n, \ell_2^n)$, the extremal points of K are precisely the unitary operators.