1. **(Duality)** Let $X$ be a Banach space. Prove that every operator $A \in L(X, X)$ satisfies:
   (i) $(\text{Im} A)^\perp = \ker A^*$;
   (ii) $(\ker A)^\perp = \text{Im} A^*$. Deduce that $\ker A = (\text{Im} A^*)_\perp$.

2. **(Fredholm’s alternative)** Prove the necessity direction in Fredholm’s theorem that we have not proved in class. Namely, let $T$ be a compact linear operator $T$ on a Banach space $X$. Prove that if $T - I$ is surjective then $T - I$ is injective. *(Hint: use the sufficiency direction in Fredholm’s theorem and the duality relations from the previous exercise.)*

3. **(Classifying the spectrum)** Compute and classify the spectrum of the following linear operators.
   (i) Multiplication operator $T$ acting on $\ell_2$ as
   
   $T((x_i)_{i=1}^\infty) = (\lambda_i x_i)_{i=1}^\infty$

   where $\lambda_i$ is a bounded sequence of complex numbers;
   (ii) Multiplication operator $T$ acting on $L_2[0, 1]$ as
   
   $(Tx)(t) = g(t)x(t)$

   where $g(t) : [0, 1] \to \mathbb{C}$ is a piecewise-continuous function (i.e. a function with finitely many points of discontinuity).

4. **(Spectrum of the adjoint operator)** Let $T \in L(X, X)$. Prove that $\sigma(T^*) = \overline{\sigma(T)}$. Here the bar stands for complex conjugation, not for closure.

5. **(Point spectrum and residual spectrum)**
   (i) Prove that if $\lambda \in \sigma_p(T)$ and $\lambda \notin \sigma_p(T^*)$ then $\lambda \in \sigma_r(T^*)$. *(Hint: use the duality relations from Exercise 1 for the operator $T - \lambda I$.)
   (ii) Prove that
   
   $\sigma_r(T) \subseteq \sigma_p(T^*) \subseteq \sigma_r(T) \cup \sigma_p(T)$.

   Deduce that if $X$ is reflexive, then $\sigma_r(T^*) \subset \sigma_p(T)$. Deduce that self-adjoint bounded linear operators in Hilbert space do not have residual spectrum.