Homework 11/13

Functional Analysis (602, Real Analysis II), Fall 2009

1. (Duality) Let X be a Banach space. Prove that every operator $A \in L(X, X)$ satisfies:

(i) $(\operatorname{Im} A)^\perp = \ker A^*$;

(ii) $(\ker A)^\perp = \operatorname{Im} A^*$. Deduce that $\ker A = (\operatorname{Im} A^*)_\perp$.

2. (Fredholm's alternative) Prove the necessity direction in Fredholm's theorem that w have not proved in class. Namely, let T be a compact linear operator T on a Banach space X. Prove that if $T - I$ is surjective then $T - I$ is injective. *(Hint: use the sufficiency direction in Fredholm's theorem and the duality relations from the previous exercise.)*

3. (Classifying the spectrum) Compute and classify the spectrum of the following linear operators.

(i) Multiplication operator T acting on ℓ_2 as

$$T((x_i)_{i=1}^\infty) = (\lambda_i x_i)_{i=1}^\infty$$

where λ_i is a bounded sequence of complex numbers;

(ii) Multiplication operator T acting on $L_2[0, 1]$ as

$$(Tx)(t) = g(t)x(t)$$

where $g(t) : [0, 1] \to \mathbb{C}$ is a piecewise-continuous function (i.e. a function with finitely many points of discontinuity).

4. (Spectrum of the adjoint operator) Let $T \in L(X, X)$. Prove that $\sigma(T^*) = \overline{\sigma(T)}$. Here the bar stands for complex conjugation, not for closure.

5. (Point spectrum and residual spectrum)

(i) Prove that if $\lambda \in \sigma_p(T)$ and $\lambda \notin \sigma_p(T^*)$ then $\lambda \in \sigma_r(T^*)$. *(Hint: use the duality relations from Exercise 1 for the operator $T - \lambda I$.)*

(ii) Prove that

$$\sigma_r(T) \subseteq \sigma_p(T^*) \subseteq \sigma_r(T) \cup \sigma_p(T).$$

Deduce that if X is reflexive, then $\sigma_r(T^*) \subset \sigma_p(T)$. Deduce that self-adjoint bounded linear operators in Hilbert space do not have residual spectrum.