**Homework 11/20**

Functional Analysis (602, Real Analysis II), Fall 2009

**1.** Consider the multiplication operator $T$ on $\ell_2$ given by

$$T((x_i)_{i=1}^{\infty}) = (\lambda_i x_i)_{i=1}^{\infty}$$

where $(\lambda_i)$ is a bounded sequence of complex numbers. Classify the sequences $(\lambda_i)$ for which $T$ is a compact operator.

**2.** Let $X$ be a Banach space, and let $S, T \in L(X, X)$. Prove that the operator $ST$ is invertible (i.e. is an isomorphism) if and only if both $S$ and $T$ are invertible. *(We used this fact in the proof of the spectral radius theorem in Lecture 30).*

**3. (Orthogonal projections)** Let $P$ be an orthogonal projection in a Hilbert space (on some proper subspace). Prove that $P$ is a self-adjoint operator. Compute and classify its spectrum.

**4. (Spectral radius of selfadjoint operators)** Let $T \in L(H, H)$ be a self-adjoint operator on a Hilbert space $H$.

(a) Prove that

$$\|T^2\| = \|T\|^2.$$

*(Hint: use Theorem on p.131).*

(b) Using part (a), prove that the spectral radius of $T$ satisfies

$$r(T) = \|T\|.$$

**5. (Inversion of perturbations)** Let $X$ be a Banach space. Show that the invertible operators (i.e. isomorphisms) on $X$ form an open set in the operator space $L(X, X)$, and the inversion map $T \mapsto T^{-1}$ is continuous on this set.

Specifically, prove that if $S \in L(X, X)$ is invertible, then every $T \in L(X, X)$ such that $\|T - S\| \leq 1/\|S^{-1}\|$ is invertible, too. *(Hint: use von Neumann's inversion lemma from Lecture 29).*