Functional Analysis (602, Real Analysis II)

Fall 2009

- Webpage
- Prerequisites: { ~~540~~ point-set topology (590), real analysis (597)

From topology: { metric spaces, compactness, completeness etc.
(→ review e.g. [BS])

From analysis: { measure theory (Lebesgue measure, integral), Hölder, Minkowski inequalities. Some Fourier analysis.

Lecture 1

I. BANACH & HILBERT SPACES

Development of analysis:

~~numbers ⟶ functions~~

numbers (before XVIIc)

↓

functions (~~differential and integral calculus~~)
(derivative, integral)

↓

sequences of functions (Weierstrass etc.; convergence Series, e.g. Taylor, Fourier) – XIX.

↓

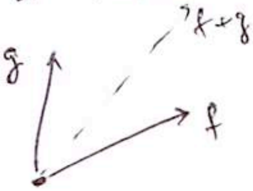
spaces of functions (functional analysis) – XX.

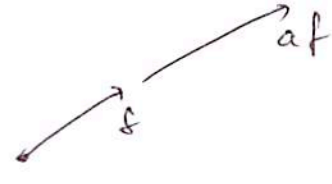
↓

? – XXI.

Why spaces of functions?

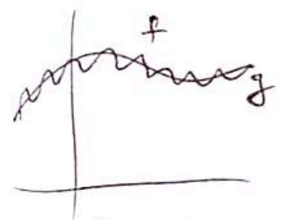
- Geometric view: look at functions as points in some "function space", or rather as vectors.



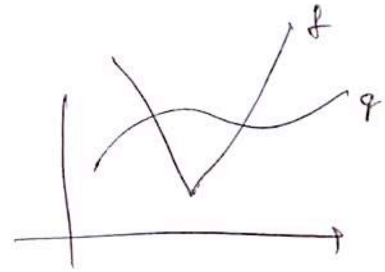


One can add functions, and multiply by scalars.
Hence a function space is a linear space.

- Further, one can envision a kind of <u>distance</u> between functions:



these two functions look "close" to each other



these not

- ~~E.g.~~ One can define a distance (metric) e.g. by

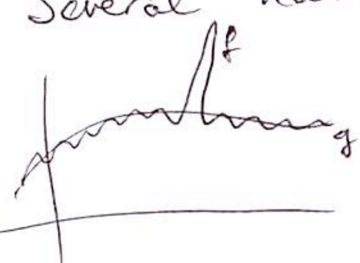
$$\rho(f,g) = \sup_x |f(x) - g(x)|.$$

~~would~~

This makes a function space <u>metric space</u>

(called <u>normed space</u>).

- ~~[illegible]~~ Several natural choices of distances:



these f and g may look "close" even though $\rho(f,g)$ is large.

$$\rho'(f,g) = \int |f(x) - g(x)| \, dx.$$

would ~~be a~~ make a better choice for a metric.

Different normed space.

- <u>Advantages of looking at function spaces</u>:

Consider <u>all</u> functions at once; soft arguments (origins - Diff Eq's)
These spaces are usually ∞-dimensional, with complic. geometry.

- <u>Historical remarks</u> (Banach & Polish school)

Linear spaces. ([Lax] is a good source.)

Examples:

1) ~~All real E =~~ $\mathbf{F}$ = {all functions ~~on $\mathbb{R}^3$~~ $\mathbb{R} \to \mathbb{R}$}. Huge.

2) $C[a,b]$ = {all continuous ~~real (or complex) valued~~ functions on $[a,b]$ → $\mathbb{R}$ or $\mathbb{C}$}.

2.5) $L_1^{[a,b]}$ = {all integrable functions on $[a,b]$}.

3) $\mathcal{P}(x)$ = {all polynomials in one variable}
$\mathcal{P}_n(x)$ = { " — " of degree $\le n$}.

3.5) $\mathcal{K}$ = {all solutions of a linear PDE in a given domain}

4) $\mathbb{R}^n$ and $\mathbb{C}^n$.

5) S = {all sequences (of real numbers)}

6) S^* = {all sequences ~~(of real numbers)~~ with finite support}

7) l_∞ = {all bounded seq}

8) c = {all convergent seq.}

9) c_0 = {all sequences converging to 0}

(1)–(3) are functions spaces,
(4) is a f. d. space
(5)–(9) are sequence spaces: ~~$S^* \subset c_0 \subset c \subset l_\infty \subset C$~~

~~Def A subspace set~~

Def (Subspace) A subset E_1 of a linear space E is a <u>subspace</u> of E if it is closed under linear operations in E.

Examples:
1) $\mathcal{P}(x) \subset C[a,b] \subset L_1[a,b] \subset \mathbf{F}$;
2) $S^* \subset c_0 \subset c \subset l_\infty \subset S$;

all subspaces.

Exercise: (a) {0} and X are linear subspaces of X
(b) Intersection of ~~all~~ $\forall$ collection of subspaces is a subspace.

From linear algebra,

Def (dimension): ~~The~~

1) Let $\{e_1, \dots, e_n\}$ be a maximal set of linearly ind. vectors in E. Then n is called the <u>dimension</u> of E ($n = \dim E$). If no such system exists, $\dim E = \infty$.
2) The dimension is independent of a particular choice of the vectors.
3) $\forall x \in E$ can be written uniquely as a linear combin.

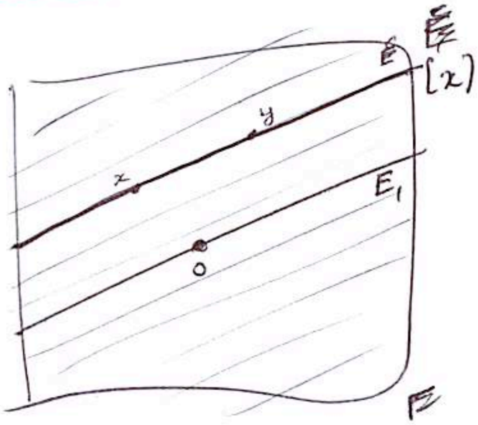
$$x = \sum_1^n a_i x_i \quad , \quad a_i \in \mathbb{R}$$

$\{e_1 \dots e_n\}$ is called a <u>basis of E</u>.

Examples:
1) F, $C(a,b)$, $L_1(a,b)$, $P(x)$ are ∞-dimensional
2) $\dim \mathcal{P}_n = n$; a basis is formed by the monomials $\{1, x, x^2, \dots x^n\}$.
3) $\dim \mathbb{R}^n = n$
4) all other sequence spaces discussed are ∞-dim.

LECTURE 2

Quotient spaces



Let $E_1 \subset E$ subspace.

- Consider the equivalence relation on E:

$$x \sim y \overset{\text{def}}{\iff} x - y \in E_1 .$$

$E/E_1 = \{$equivalence classes (or "cosets") $[x]$ of all $x \in E\}$.

- Observe that $[x] = x + E_1 = \{x + h,\ h \in E_1\}$.

We can make E/E_1 into a linear space by defining ~~the~~

$$[x]+[y] := [x+y]$$
$$a[x] = [ax].$$

Def: E/E_1 is called the "<u>quotient space</u>".
$\dim E/E_1$ is called the <u>codimension</u> of E $(=: \operatorname{codim} E)$.

~~Fact (from linear algebra) If E is finite dimensional, $E \subset E_1 \Rightarrow$~~
~~$\dim E = \dim E_1 + \operatorname{codim} E_1$~~

~~Examples 1) Consider $c_0 \subset c$.~~
~~Then $\operatorname{codim}(c_0) = 1$~~
~~$\forall x \in c,$~~

<u>Examples</u> 1) $L_1[a,b]$ is formally defined as the quotient space E/E_1 of ~~all~~ $E = \{$all integrable functions$\}$ over $E_1 = \{$all functions supported on measure zero sets$\}$.
— allows to identify functions up to negligible sets.

2) $c_0 \subset c$. Then $\operatorname{codim} c_0 = 1$

~~$\forall x \in c$; $x = y + z$, $y \in c_0$, z = constant sequence~~ ~~$x = a\mathbb{1} + z$~~ $z \in c_0$

~~$x = y + \mathbb{1} \cdot a$~~ ~~$\forall x, y \in$~~

$\Rightarrow$ ~~$c/c_0 =$~~

$\forall x \in c: \quad x = a\mathbb{1} + z; \quad a \in \mathbb{R}, z \in c_0$
uniquely

$\Rightarrow [x] = a[\mathbb{1}] + \underset{0}{\underset{\shortparallel}{[z]}} = a[\mathbb{1}].$

Linear operators

Def A map $A: E \to F$ between two linear spaces E and F is ~~a linear~~ called a linear operator ~~(linear map)~~ (transformation, map)

if $$A(ax + by) = a\,A(x) + b\,A(y) \qquad \forall x, y \in E, \ \forall a, b \in \mathbb{R}$$
often written as $aAx + bAy$

Def $\ker(A) = \{x \in E : Ax = 0\}$
$\operatorname{Im}(A) = \{Ax : x \in E\}$.

Exercise: A is one-to-one $\Leftrightarrow$ $\ker A = \{0\}$.

Examples (a) $A(f) = f'$, $A: \mathcal{P}(x) \to \mathcal{P}(x)$.

(b) Embedding ~~operator~~. $E_1 \subseteq E$

$A: E_1 \to E \ : \ Ax = x$

(c) Quotient map : $E_1 \subseteq E$

$A: E \to E/E_1 \ : \ Ax = [x]$

Exercise: check linearity, show that $\ker A = E_1$, $\operatorname{Im} A = E/E_1$ (onto).

(d) Shift on sequence space : $(Af)(k) = f(k+1)$, $\forall f, \forall k \in \mathbb{Z}$