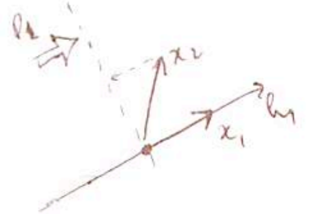
LECTURE 10

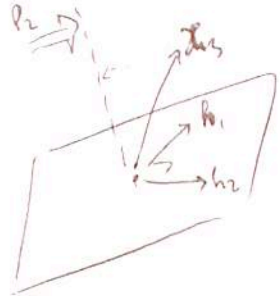
Gram-Schmidt Orthogonalization. Isometry ~~of all~~ btw Hilb. spaces

~~Thm (Gram-Schmidt)~~. Let $(x_k)_1^\infty$ be a linearly independent system in a Hilbert space X. ~~There exists~~

Want: to transform (x_k) into an orthonormal system (h_k).

Here's how (Gram-Schmidt orthogonalization procedure)





$$h_1 = \frac{x_1}{\|x_1\|}$$

$$h_2 = \frac{P_1 x_2}{\|P_1 x_2\|} \quad \text{where } P_1 \text{ is the orthog. proj. onto } \operatorname{span}(h_1)^\perp$$

$$h_3 = \frac{P_2 x_3}{\|P_2 x_3\|}, \quad \text{where } P_2 \text{ is the orthog. proj onto } \operatorname{span}(h_1, h_2)^\perp$$

$\dots$

$$h_{n+1} = \frac{P_n x_{n+1}}{\|P_n x_{n+1}\|}, \quad \text{where } P_n \text{ is the orthog. proj. onto } \operatorname{span}(h_1, \dots, h_n)^\perp$$

$\dots$

How to compute P_n?

~~By the orthogonal decomposition $X = \operatorname{span}(h_1 \dots h_n) \oplus \operatorname{span}(h_1 \dots h_n)^\perp$,~~
~~the orth. proj. onto~~

By ~~Thm (optimality) p.39~~ (Cor p 39), the orthogonal projection Q_n onto $\operatorname{span}(h_1 \dots h_n)$
has the form $Q_n x = \sum_{k=1}^{n} (x, h_k) h_k$ (Fourier series of x)

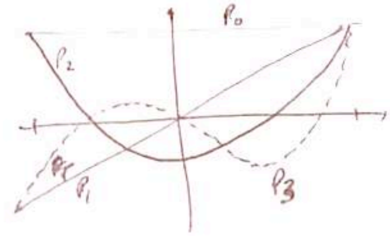
Hence $P_n x = x - Q_n x = x - \sum_{k=1}^{n} (x, h_k) h_k$.

$$\Rightarrow \boxed{h_{n+1} = \frac{x - \sum_{k=1}^{n} \langle x, h_k \rangle h_k}{\| x - \sum_{k=1}^{n} \langle x, h_k \rangle h_k \|}, \quad k = 0, 1, 2, \dots}$$

Example: Orthogonal polynomials.

1) Apply Gram-Schmidt orthogonalization to the monomials $(t^k)_{k=0}^{\infty}$ in $L_2[-1,1]$ $\Rightarrow$ get <u>Legendre polynomials</u> (up to normalization):

$$P_0(t) = 1,$$
$$P_1(t) = x$$
$$P_2(t) = \tfrac{1}{2}(3t^2 - 1)$$
$$P_3(t) = \tfrac{1}{2}(5t^3 - 3x)$$
$$\cdots$$
$$P_k(t) = \frac{1}{2^k k!} \frac{d^k}{dt^k} (t^2 - 1)^k.$$
$$\cdots$$



(Ex.)

Similar constructions can be done for L_2 <u>with weight</u>, i.e. $L_2([a,b], w(t)dt)$ where $w(t) \geq 0$ is a weight function.
i.e. $\langle f, g\rangle = \int f \bar{g}\, w\, dt$

2) For example, for $w(t) = e^{-t^2}$, G-S orthogonalization of the monomials ~~produces~~ in $L_2(-\infty, \infty)$

<u>Hermite polynomials</u>

$$H_k(x) = (-1)^k e^{t^2} \frac{d^k}{dt^k} e^{-t^2}$$

3) Other examples: Chebyshev, Jacobi, Legendre polynomials.
[see MathWorld].

THM (Gram - Schmidt)

> Let $(x_k)_1^\infty$ be a linearly independent system in a Hilbert space X.
> There exists an orthonormal system $(h_k)_1^\infty$ in X such that
> $$\text{span}\,(h_k)_1^n = \text{span}\,(x_k)_1^n \quad \text{for all } n \in \mathbb{N}.$$

~~Consequently~~,

(h_k) is orthonormal by construction.
Also by construction, the dimensions of the spaces $\text{span}\,(h_k)_1^n$ and $\text{span}\,(x_k)_1^n$ are equal. QED -

It follows that $\overline{\text{span}}\,(h_k)_1^\infty = \overline{\text{span}}\,(x_k)_1^\infty$. $\Rightarrow$

Cor $\boxed{\text{Every separable Hilbert space has an orthonormal basis.}}$

- Recall that a metric space is ~~called~~ separable if it contains a countable dense subset.

- In other words, X is separable $\Leftrightarrow$ $\exists$ complete $(x_k)_1^\infty \subset X$
  $\Rightarrow$ $\overline{\text{span}}\,(h_k)_1^\infty = \overline{\text{span}}\,(x_k)_1^\infty = X$.

- Examples of ~~dense~~ separable spaces:

  1) $C[0,1]$ (polynomials with rational coefficients)

  2) $L_2[0,1]$ (trigonometric poly. with rational coeff's - see p (41))

  3) $\ell_2[0,1]$, (sequence with finite support and with rational coeff)

  4) ℓ_p $(1 \le p < \infty)$, c_0 (same as in (3))

  Example of non-separable space:

  5) ℓ_∞ (Ex.)

~~Examples of~~ ~~orthogo~~ orthonormal bases ~~(constructed by G-S orthogonalization)~~

THM (Isometry) All separable infinite-dimensional Hilbert spaces are isometric.

Precisely, for every two separable Hilbert spaces X and Y, there exists a linear ~~onto~~ bijective map $T : X \to Y$ s.t.

$$\langle Tx, Ty \rangle = \langle x, y \rangle \qquad \forall x, y \in X$$

Remark $\Rightarrow \|Tx\| = \|x\| \quad \forall x$

$\Rightarrow \|Tx - Ty\| = \|x - y\| \quad \forall x, y \in X.$

(preserves distances).

WLOG, $Y = \ell_2$ (for then one can factor through ℓ_2: $X \to \ell_2 \to Y$)

Let $(x_k)_1^\infty$ be an orthonormal basis in X.

Define $T : X \to \ell_2$ as

$$Tx = \left(\langle x, x_k \rangle \right)_{k=1}^\infty ,$$

i.e. we assign $\forall x$ ~~thhhh~~ the sequence of its Fourier coefficients.

~~By Bessel's ineq~~ By Parseval's identity,

$$\|Tx\| = \|x\|. \qquad (*)$$

(in particular T indeed maps into ℓ_2).

~~T has~~ The inverse map is

$$T^{-1}\left((a_k)_1^\infty \right) = \sum_{k=1}^\infty a_k x_k$$

(Indeed, this map is well defined by the convergence criterion for ~~orthogonal orthonormal~~ orthogonal series (Thm p. 35). Also, $T^{-1} T x = \sum_{k=1}^\infty \langle x, x_k \rangle x_k = x$ by Fourier expansion.)

Hence T is a linear bijection which preserves the norm $(*)$.

~~Therefore, T preserves the inner~~

Since the inner product is determined by the norm (by polarization identity – see HW3), T preserves the inner product as well.

QED.