Definition and examples. Norm.

**Def.** Let $E$ be a linear space. A linear operator $f : E \rightarrow \mathbb{R}$ (or $\mathbb{C}$) are called linear functionals on $E$. Equivalently, a function $f : E \rightarrow \mathbb{R}$ (or $\mathbb{C}$) is a linear functional if
\[ f(ax + by) = af(x) + bf(y) \quad \text{for all } x, y \in E; \quad a, b \in \mathbb{R} \text{ (or } \mathbb{C}) \]

**Examples.**

1) On $E = \mathbb{R}^n$,
\[ f(x) = \sum_{i=1}^n y_i x_i, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n \]
where $y_i$ are fixed numbers.

2) More generally, on a Hilbert space $E = X$,
\[ f(x) = \langle x, y \rangle, \quad x \in X, \]
where $y \in X$ is a fixed vector.

3) On $E = C([0, 1])$,
\[ f(g) = \int_0^1 x(t) g(t) \, dt, \quad x \in L^1([0, 1]) \]
where $g \in C([0, 1])$ is a fixed function.
4) "Dirac delta function" on $C([0,1])$ is the functional $f(x) = x(t_0), \quad x \in C([0,1])$.

This gives rigorous meaning to the "integral"

$$\int_0^1 x(t) \delta_{t_0}(t) \, dt = x(t_0).$$

**Def (Boundedness, continuity):**

Let $f$ be a linear functional on a normed space $X$.

1. $f$ is continuous if $f$ is a constant
   - $x_n \to x$ implies $f(x_n) \to f(x)$.
2. $f$ is called bounded if $\exists C$ s.t.
   - $|f(x)| \leq C \|x\|$ for all $x \in X$.

**Prop** $f$ is continuous if $f$ is bounded.

(\Rightarrow) If $f$ is not bounded, then $\exists (x_n) \subseteq X$ s.t.

$$|f(x_n)| \geq n \|x_n\|, \quad n = 1, 2, \ldots$$

$$\Rightarrow |f\left(\frac{x_n}{n\|x_n\|}\right)| \geq 1, \quad n = 1, 2, \ldots$$

On the other hand, $\frac{x_n}{n\|x_n\|} \to 0$ (its norm is $\frac{1}{n}$).

This contradicts the continuity of $f$. \(\Box\)

(\Leftarrow) Let $x_n \to x$. Then

$$|f(x_n) - f(x)| = |f(x_n - x)| \leq C \|x_n - x\| \to 0.$$

\(\Box\)

Exercise: If $f$ is continuous at a single point then $f$ is continuous (everywhere)
The linear space of all continuous linear functionals on \( X \) is called the **dual space** \( X^* \).

\( X^* \) is a normed space with the norm defined as

\[
\|f\| := \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} = \sup_{\|x\|=1} |f(x)|, \quad f \in X^*.
\]

**Exercise:**
1. Check the identity

\[
\|f(x)\| \leq \|f\| \cdot \|x\| \quad \forall x \in X, \ f \in X^*.
\]

and \( \|f\| \) is the best constant in this inequality.

2. \( X^* \) is always a Banach space (i.e. complete), even if \( X \) is incomplete.

Will prove this later as a more general statement about linear operators.

**Exercise:** \( \ker f \) has codimension \( 1 \); such subspaces are called hyperplanes.
Prop (hyperplanes) Let \( f \) be a linear functional on a linear space \( E \). Then:

(i) \( \ker f \) is a hyperplane in \( E \), i.e., \( \dim (\ker f) = 1 \).

(ii) If \( g \) is another linear functional on \( E \), then \( \ker f = \ker g \) implies \( f = ag \) for some \( a \neq 0 \).

(iii) For every hyperplane \( H \subseteq E \), there exists a linear functional \( f+o \) on \( E \) such that \( \ker f = H \).

It is enough to find a 1-dimensional linear subspace \( F \subseteq E \) s.t. \( \ker f \cap F = \{0\} \), \( \ker(\cdot + F) \subseteq E \).

\[ \exists x_0 : f(x_0) \neq 0; \quad F = \text{span} \{x_0\} \]

By considering \( x_0 / f(x_0) \), w.l.o.g. we can assume \( f(x_0) = 1 \). It is clear that \( \ker f \cap F = \{0\} \).

To prove \( F \subseteq \ker f \), let \( x \in E \). Then

\[
\begin{align*}
\mathbf{x} &= f(x) x_0 + (x - f(x) x_0) \\
&= f(x) \frac{x_0}{f(x_0)} + (x - f(x) x_0) \\
&= f(x) \frac{x_0}{f(x_0)} + (x - f(x) x_0)
\end{align*}
\]

\[ \tag{1} \]

\[ F \subseteq \ker f \]

(ii) Applying \( g \) to both sides of (1), we obtain

\[ g(x) = f(x) g(x_0) + 0 \]

Clearly \( a \neq 0 \) (for otherwise \( g = 0 \Rightarrow \ker g = \ker E = \ker f = E \Rightarrow f = 0 \) \).

(iii) \( \dim (E/H) = 1 \Rightarrow E/H = \{ a [x_0] : a \in \mathbb{R} \} \).

\[ \Rightarrow \] every \( [x] \in E/H \) can be represented \( (x) = a [x_0] \) for some \( a \in \mathbb{R} \).

\[ \Rightarrow \] if \( x \in E \), \( x = a x_0 + h \), \( a \in \mathbb{R} \), \( h \in H \).

Define \( f \) by \( f(x) = a \). Then \( \ker f = H \) and clearly \( \ker f = H \).

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