# LECTURE 11

## Linear Functionals

### Definition and examples. Norm.

**Def** Let $E$ be a linear space (over $\mathbb{R}$ or $\mathbb{C}$). ~~A linear operato~~
Linear operators $f : E \to \mathbb{R}$ (or $\mathbb{C}$) are called <u>linear functionals on $E$</u>.
Equivalently, a function $f : E \to \mathbb{R}$ (or $\mathbb{C}$) is a linear functional if

$$f(ax + by) = af(x) + bf(y) \quad \text{for all } x, y \in E;\ a, b \in \mathbb{R} \text{ (or } \mathbb{C}).$$

**Examples** 1) On $E = \mathbb{R}^n$,

$$f(x) := \sum_{i=1}^{n} x_i y_i \quad , \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n$$

where $y_i$ are fixed numbers.

2) More generally, on a Hilbert space ~~H~~ $X$,

$$f(x) := \langle x, y \rangle \quad , \quad x \in X,$$

where $y \in X$ is a fixed vector

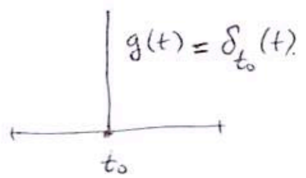
~~3) On $C[0,1]$, $f(x) = x(t_0)$, $x \in C[0,1]$ where $t_0 \in [0,1]$ is a fixed number.~~

3) On $C[0,1)$,

$$F(x) = \int_0^1 x(t)\, g(t)\, dt \quad , \quad x \in L_\infty(0,1)$$

~~[illegible]~~

where $g \in L_1[0,1]$.




4) "<u>Dirac delta function</u>":
on $C[0,1]$
is the functional

$$f(x) = x(t_0), \quad x \in C[0,1].$$

This gives rigorous meaning to the "integral"

$$\int_0^1 x(t)\, \delta_{t_0}(t)\, dt = x(t_0).$$

<u>Def</u> (<u>Boundedness, continuity</u>).

Let $f$ be a linear functional on a normed space $X$.

Recall that

1) $f$ is <u>continuous</u> if ~~it is a contin~~

$$x_n \to x \quad \text{implies} \quad f(x_n) \to f(x).$$

2) $f$ is called <u>bounded</u> if $\exists\, C$ s.t.

$$|f(x)| \leq C \|x\| \quad \text{for all } x \in X.$$

<u>Prop</u> $\boxed{f \text{ is continuous iff } f \text{ is bounded.}}$

($\Rightarrow$) Suppose $f$ is not bounded. Then $\exists\, (x_n) \subset X$ s.t.

$$|f(x_n)| \geq n \|x_n\|, \quad n = 1, 2 \ldots$$

$$\Rightarrow \left| f\left( \frac{x_n}{n\|x_n\|} \right) \right| \geq 1, \quad n = 1, 2 \ldots$$

On the other hand, $\dfrac{x_n}{n\|x_n\|} \to 0$ (its norm is $\frac{1}{n}$).

This contradicts the continuity of $f$. $\lightning$

($\Leftarrow$) Let $x_n \to x$. Then

$$|f(x_n) - f(x)| = |f(x_n - x)| \leq C \|x_n - x\| \to 0.$$

QED.

✓ <u>Exercise</u> If $f$ is continuous at a single point then $f$ is continuous (everywhere).

/Prop.

Def (Dual space).

The linear space of all continuous linear functionals on $X$ is called the dual space $X^*$.

$X^*$ is a normed space with the norm defined as

$$\|f\| := \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} = \sup_{\|x\|=1} |f(x)|, \qquad f \in X^*.$$

Exercise: 1) Check the identity

2) Check the norm axioms on $X^*$ based on .

Remarks 1) We always have the bound

$$\boxed{|f(x)| \leq \|f\| \cdot \|x\| \qquad \forall x \in X,\ f \in X^*}$$

and $\|f\|$ is the best const. in this inequality.

2) $X^*$ is always a Banach space (i.e. complete), even if $X$ is incomplete.

Will prove this later as a more general statement about linear operators.

~~Exercise: $\ker f$ has codimension 1; such subspaces are called hyperplanes.~~

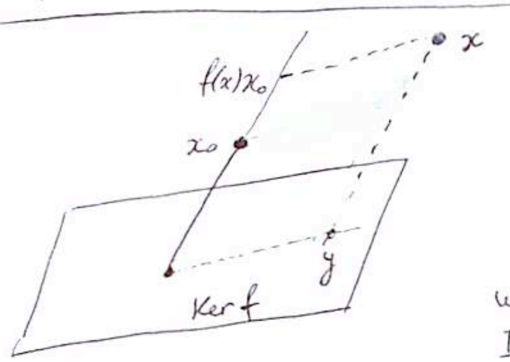
**Prop** (*hyperplanes*) Let $f \neq 0$ be a linear functional on a linear space $E$. Then:

(i) $\ker f$ is a *hyperplane* in $E$, i.e. $\operatorname{codim}(\ker f) = 1$.

(ii) If $g \neq 0$ is another linear functional on $E$, then

$$\ker f = \ker g \quad \text{implies} \quad f = ag \quad \text{for some } a \neq 0$$

(iii) For every hyperplane $H \subseteq E$, there exists a linear functional $f \neq 0$ on $E$ such that $\ker f = H$.




(i) It is enough to find a 1-dimensional linear subspace $F$ of $E$ s.t. $\ker f \cap F = \{0\}$, $\ker f + F = E$.

$\exists\, x_0 : f(x_0) \neq 0$; $F := \operatorname{span}(x_0)$.

By considering $x_0/f(x_0)$, W.L.O.G. we can assume $f(x_0) = 1$.

It is clear that $\ker f \cap F = \{0\}$.

To prove $F + \ker f = E$, let $x \in E$. Then

$$(*) \qquad x = \underbrace{f(x)x_0}_{\in F} + \underbrace{(x - f(x)x_0)}_{\in \ker f}. \qquad \text{QED}$$

(ii) Applying $g$ to both sides of $(*)$, we obtain

$$g(x) = f(x)\underbrace{g(x_0)}_{= a} + 0$$

Clearly $a \neq 0$ (for otherwise $g = 0 \Rightarrow \ker g = E \Rightarrow \ker f = E \Rightarrow f = 0$ ↯).

(iii) $\dim(E/H) = 1 \Rightarrow E/H = \{a[x_0] : a \in \mathbb{R}\}$

$\Rightarrow$ every $[x] \in E/H$, ~~can be repres~~ $[x] = a[x_0]$ for some $a \in \mathbb{R}$

$\Rightarrow \forall\, x \in E, \quad x = ax_0 + h, \quad a \in \mathbb{R},\ h \in H.$

Define $f$ by $f(x) = a$. Then ~~$\ker f = H$~~ $f$ is a linear functional (check!) and clearly $\ker f = H$.