LECTURE 12

Linear functionals on classical spaces

Riesz Representation Thm.

Hilbert space

THM (Riesz Representation Thm.) Let X be a Hilbert space.

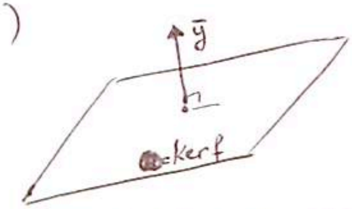
(i) For every vector $y \in X$, the function
$$f(x) = \langle x, y \rangle, \qquad x \in X,$$
is a bounded linear functional on X, with the norm $\|f\| = \|y\|$.

(ii) Conversely, for every bounded linear functional f on X, there exists a unique vector $y \in X$ such that
$$f(x) = \langle x, y \rangle, \quad x \in X.$$
Moreover, this correspondence is an isometry: $\|f\| = \|y\|$.

Remark In other words, $X^* = X$.

(i) Cauchy - Schwartz $\Rightarrow$
$$|f(x)| \le \|x\| \|y\|, \quad x \in X$$
$\Rightarrow$ ~~$\|f\| \le \|y\|$~~ $\Rightarrow f \in X^*$ and $\|f\| \le \|y\|$.

For $x = y$, $f(x) = \langle x, x\rangle = \|x\|^2 \Rightarrow$
$\|f\| \ge \|y\|$. QED

(ii) 

Let $f \in X^*$. By Prop. on hyperplanes (p. 49), $\ker f$ is a hyperplane in X ~~$\Rightarrow$ by the ortho~~
~~$\Rightarrow$ we can write an orthogonal $\Rightarrow \exists\, \bar{y} \in (\ker f)^\perp$~~

$\Rightarrow$ one can write an orthogonal decomposition
$$X = H \oplus \operatorname{span}(\bar{y}) \qquad \text{for some } \bar{y} \in (\ker f)^\perp$$

Consider the linear functional
$$g(x) := \langle x, \bar{y}\rangle, \quad x \in X.$$

Clearly, $\ker g = (\bar{y})^\perp = \ker f$. By Prop. on hyperplanes (p. 49), $f = ag$
$$f(x) = a\langle x, \bar{y}\rangle = \langle x, \bar{a}\bar{y}\rangle. \Rightarrow y := \bar{a}\bar{y} \quad \text{QED}$$

Example of application:

(see e.g. Lax, Functional Analysis, 7.1)

<u>Von Neumann's proof of Radon-Nikodym Thm</u>

- let μ, ν be measures on the same σ-algebra.

Recall that ν is called <u>absolutely continuous</u> w.r.to μ

(written $\nu \ll \mu$) if

$$\mu(A) = 0 \quad \text{implies} \quad \nu(A) = 0.$$

<u>THM</u> (<u>Radon-Nikodym</u>) If $\nu \ll \mu$ then ν can be expressed as

$$\nu(A) = \int_A g \, d\mu,$$

where $g \geq 0$ is an integrable function w.r.to μ. $\left(g =: \frac{d\nu}{d\mu}\right)$.

(For finite measures).

The linear functional

$$F(f) := \int f \, d\mu$$

is bounded on $L_2(\mu)$, hence also on $L_2(\mu+\nu)$.

By Riesz Repr. Thm, $\exists \, h \in L_2(\mu+\nu)$ s.t.

$$(*) \quad \boxed{\int f \, d\mu = \int f h \, d(\mu+\nu)} \quad \text{for all } f \in L_2(\mu+\nu)$$

~~By the Monotone Convergence Thm, (*) holds~~

<u>Claim</u>: $0 < h \leq 1$ μ-a.e.

1) Consider the set $B := \{h \leq 0\}$ and the function $f := \mathbb{1}_B$.

Then (*) becomes $\mu(B) = \int_B h \, d(\mu+\nu) \leq 0 \quad \Rightarrow \mu(B) = 0.$

2) Consider the set $B := \{h > 1\}$ and the function $f := \mathbb{1}_B \Rightarrow$

$$\mu(B) = \int_B h \, d(\mu+\nu) \geq (\mu+\nu)(B) \quad \Rightarrow \mu(B) = 0.$$

By Monotone Convergence Thm, (*) holds for all $f \geq 0$.

Rewrite (*) as

$$\boxed{\int f h \, d\nu = \int f(1-h) \, d\mu \, , \qquad f \geq 0.}$$

Now, for a set A, choose $fh := \mathbb{1}_A$

$\Rightarrow f := \mathbb{1}_A / h$.

Hence
$$\nu(A) = \int_A \frac{1-h}{h} \, d\mu.$$

The proof is complete with $g = \frac{1-h}{h}$.

Remark Lebesgue Decomposition Thm can be proved similarly.

LECTURE 13

Linear functionals on other classical spaces.

- Riesz Repr. Thm $\Rightarrow$ 1) $(\ell_2)^* = \ell_2$ via
$$f(x) = \langle x, y \rangle = \sum_1^\infty x_i \bar{y}_i \, ,$$
where $y = (y_i) \in \ell_2$ is a fixed vector.
2) $L_2^* = L_2$ via $F(f) = \langle f, g \rangle = \int f \bar{g} \, d\mu$, where $g \in L_2$ is a fixed function

- Generalization for ℓ_p:

THM ($\ell_p^* = \ell_{p'}$). For $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$,

we have $\ell_p^* = \ell_{p'}$.

As in Riesz Representation Thm, this means that:

(i) For every $y \in \ell_{p'}$, the function
$$f(x) = \sum_1^\infty x_i y_i \, , \quad x \in X,$$
is a bounded linear functional on ℓ_p, with the norm $\|f\| = \|y\|_{p'}$.

(ii) Conversely, for every bounded linear functional f on ℓ_p, there exists a unique $y \in \ell_p$ s.t.
$$f(x) = \sum_1^\infty x_i y_i, \quad x \in X.$$
Moreover, $\|f\| = \|y\|_{p'}$.

Remark: Implies R.R.T.