Rewrite (*) as

$$\boxed{\int f h \, d\nu = \int f(1-h)\, d\mu, \qquad f \geq 0.}$$

Now, for a set A, choose $fh := \mathbb{1}_A$

$\Rightarrow f = \mathbb{1}_A / h$.

Hence
$$\nu(A) = \int_A \frac{1-h}{h}\, d\mu.$$

The proof is complete with $g = \frac{1-h}{h}$.

<u>Remark</u> Lebesgue Decomposition Thm can be proved similarly.

LECTURE 13

Linear functionals on other classical spaces.

- Riesz Repr. Thm $\Rightarrow$ 1) $(\ell_2)^* = \ell_2$ via

$$f(x) = \langle x, y \rangle = \sum_1^\infty x_i \bar{y}_i,$$

where $y = (y_i) \in \ell_2$ is a fixed vector.

2) $L_2^* = L_2$ via $F(f) = \langle f, g \rangle = \int f \bar{g}\, d\mu$, where $g \in L_2$ is a fixed function

- Generalization for ℓ_p:

<u>THM</u> ($(\ell_p)^* = \ell_{p'}$). For $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$,

we have $\ell_p^* = \ell_{p'}$.

As in Riesz Representation Thm, this means that:

(i) For every $y \in \ell_{p'}$, the function
$$f(x) = \sum_1^\infty x_i y_i, \quad x \in X,$$
is a bounded linear functional on ℓ_p, with the norm $\|f\| = \|y\|_{p'}$.

(ii) Conversely, for every bounded linear functional f on ℓ_p, there exists a unique $y \in \ell_p$ s.t.
$$f(x) = \sum_1^\infty x_i y_i, \quad x \in X.$$
Moreover, $\|f\| = \|y\|_{p'}$.

<u>Remark</u>: Implies R.R.T.

(i) follows from Hölder's inequality:

$$\left|\sum_i x_i y_i\right| \le \left(\sum |x_i|^p\right)^{1/p} \left(\sum |y_i|^{p'}\right)^{1/p'}$$ ~~equality iff $x_i = e^{-i\operatorname{Arg}(y_i)} |y_i|^{p'-1}$~~

$$\Leftrightarrow \quad |f(x)| \le \|x\|_p \, \|y\|_{p'}$$

$\Rightarrow \|f\| \le \|y\|_{p'}$. To prove the reverse, ~~inspect the~~ note that

$\exists$ Equality case in Hölder inequality:

$$\left|\sum x_i y_i\right| = \|x\|_p \, \|y\|_{p'} \qquad \left(\text{for} \quad x_i = e^{-i\operatorname{Arg}(y_i)} |y_i|^{p'-1}\right).$$

QED.

(ii) Let $f \in \ell_p^*$. The linear functional is determined by its action on the coord. basis $e_i = (0, 0, \dots, 0, \underset{k}{1}, 0, \dots)$.

Then
$$f(x) = f\left(\sum_1^\infty x_i e_i\right) = \sum_1^\infty x_i f(e_i) = \sum_1^\infty x_i y_i, \qquad x \in \ell_p,$$

where $y = (y_i) = (f(e_i))$.

To show that $y \in \ell_{p'}$, use the equality case of Hölder's ineq:

$$\|f\|\,\|x\|_p \ge |f(x)| = \left|\sum_1^\infty x_i y_i\right|$$

$$= \|x\|_p \, \|y\|_{p'}$$

$$\Rightarrow \|y\|_{p'} \le \|f\|.$$

The reverse inequality was proved in (i).

Exercises: 1) $c_0^* = \ell_1$

2) $\ell_1^* = \ell_\infty$.

THMS $L_p^* = L_{p'}$ $(1 \le p < \infty)$,

$C(K)^* = \{$regular Borel signed measures on $K\}$

$L_\infty(\Omega)^* = \{\sigma$-additive bounded signed measures on $\Omega\}$.

Extensions of functionals.

Extensions of continuous linear functional from a subspace X_0 to the whole space X ?
$f_0 \in X_0^* \longrightarrow f \in X^*$ s.t. $f_0|_{X_0} = f|_{X}$ (i.e. $f_0(x) = f(x)$, $x \in X_0$).

Easy when $X_0 \subseteq X$ dense:

Prop (Extension by continuity) Let $X_0 \subseteq X$ be a dense subspace of a normed space X. Then every $f_0 \in X_0^*$ admits a unique extension $f \in X^*$. Moreover, $\|f\| = \|f_0\|$.

Example of applications: (Lebesgue meas.) $C[0,1] \subset X$ where X is the completion of $C[0,1]$. $F(f) = \int f(t)\,dt$. Extension – Lebesgue int; $X := L_1[0,1]$.

By density, $\forall x \in X \ \exists (x_n) \subset X_0 : x_n \to x$.

Then $(f_0(x_n))$ is a Cauchy sequence:

$$|f_0(x_n) - f_0(x_m)| \le \|f_0\| \cdot \|x_n - x_m\| \to 0.$$

Denote the limit of this sequence $f(x)$.

Then:

1) $f(x)$ is well defined, i.e. ~~does not~~ depends only on x and not on the choice of sequence $x_n \to x$.
Indeed, if $x_n' \to x$ then

$$|f_0(x_n) - f_0(x_n')| \le \|f_0\| \, \|x_n - x_n'\| \to 0,$$

hence the limits of $f_0(x_n)$ and of $f_0(x_n')$ coincide.

2) f is a linear functional; if $x_n \to x$, $y_n \to y$ $\Rightarrow$

$$f(ax + by) = \lim_n f_0(a x_n + b y_n) = a \lim_n f_0(x_n) + b \lim_n f_0(y_n)$$
$$= af(x) + bf(y).$$

3) f is bounded: if $x_n \to x$ then

$$|f(x)| = \lim_n |f_0(x_n)| \le \|f_0\| \lim \|x_n\| = \|f_0\| \, \|x\|.$$

Then $f \in X^*$, $\|f\| \le \|f_0\|$. ~~However~~
Since f is an extension of f_0, we also have $\|f\| \ge \|f_0\|$.

4). Uniqueness: two continuous functions which coincide on a dense set, ~~coincid~~ are equal.