LECTURE 14

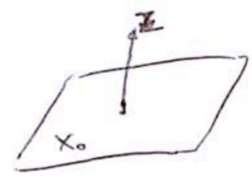
Hahn-Banach Theorem

Extensions from proper closed subspaces.

Hahn-Banach Thm (Analytic form) Let X_0 be a closed subspace of a normed space X. Then every $f \in X_0^*$ admits an extension $f \in X^*$ such that

$$\|f\| = \|f_0\|.$$

1) Extension by one dimension: Assume first that X_0 is a hyperplane in X, i.e. $\operatorname{codim} X_0 = 1$. ~~Therefore $\exists\, x \in X$~~



Fix $z \in X \setminus X_0$; then every vector $x \in X$ can be uniquely represented as

$$x = az + x_0, \quad a \in \mathbb{R},\ x_0 \in X_0$$

~~Since f_0 is~~ $\Rightarrow f(x) = af(z) + f(x_0) = af(z) + f_0(x_0)$

~~so f is de~~
So the desired extension f is determined by just one number $\underline{f(z) =: C}$.

W.L.O.G. $\|f_0\| = 1$, i.e. $|f_0(x_0)| \le \|x_0\| \ \forall x_0 \in X_0$.

We are looking for $\|f\| = 1$, i.e. $|f(x)| \le \|x\| \ \forall x \in X$.

$$\Updownarrow$$

$$f(x) \le \|x\| \quad \forall x \in X.$$

$$\Leftrightarrow \boxed{aC + f_0(x_0) \le \|az + x_0\| \quad \forall a \in \mathbb{R},\ x_0 \in X_0} \quad (*)$$

For $a = 0$ (*) is true by the assumption.

For $a > 0$ and $a < 0$ ($b := -a$): ~~equivalent to This is equivalent to:~~

~~For $a > 0$, the (*) is equivalent~~

$$C \le \left\|z + \frac{x_0}{a}\right\| - f_0\left(\frac{x_0}{a}\right), \quad a > 0,\ x_0 \in X_0.$$

$$C \ge f_0\left(\frac{x_1}{b}\right) - \left\|\frac{x_1}{b} - z\right\|, \quad b > 0,\ x_1 \in X_1.$$

Existence of such C is equivalent to

$$\sup_{b>0,\, x_1\in X_0}\left(f_0\left(\frac{x_1}{b}\right) - \left\|\frac{x_1}{b} - z\right\|\right) \le \inf_{a>0,\, x_0\in X_0}\left(\left\|z+\frac{x_0}{a}\right\| - f_0\left(\frac{x_0}{a}\right)\right),$$

i.e. to the inequality

$$f_0\left(\frac{x_1}{b}\right) - \left\|\frac{x_1}{b} - z\right\| \le \left\|z+\frac{x_0}{a}\right\| - f_0\left(\frac{x_0}{a}\right), \quad a,b>0;\ x_0,x_1\in X_0.$$

$$(\Leftrightarrow) \quad f_0\left(\frac{x_0}{a}+\frac{x_1}{b}\right) \le \left\|z+\frac{x_0}{a}\right\| + \left\|\frac{x_1}{b} - z\right\|.$$

$\vee$ (Δ inequality)

This is true:

$$f_0\left(\frac{x_0}{a}+\frac{x_1}{b}\right) \le \left\|\frac{x_0}{a}+\frac{x_1}{b}\right\| .$$

QED.

2). Transfinite induction.

Recall Zorn's Lemma: A partially ordered set in which every chain has an upper bound contains a maximal element.

~~Consider the set Γ of pairs (Y, g) where $X_0 \subseteq Y \subseteq X$~~

~~Y is a subspace, $X_0 \subseteq Y \subseteq X$~~

Consider the set of all extensions of f_0.

Precisely, let Γ be the set of pairs (Y,g) where Y is a subspace, $X_0 \subseteq Y \subseteq X$, and $g \in Y^*$ is an extension of f_0.

We want to show that Γ contains an element with $Y=X$.

Consider the partial order on Γ:

$$(Y_1,g_1) \le (Y_2,g_2) \text{ if } Y_1 \subseteq Y_2,\ g_2 \text{ is an extension of } g.$$

Then every chain $((Y_\alpha, g_\alpha))_\alpha$ has an upper bound (Y,g) in Γ:

$$Y := \bigcup_\alpha Y_\alpha, \quad g(x) := g_\alpha(x) \text{ if } x \in Y_\alpha.$$

Hence, by Zorn's lemma, $\exists$ a maximal element (Y,g) in Γ.

by part (1) of the proof, we could extend g onto

Consequences of Hahn-Banach Theorem.

X: normed space.

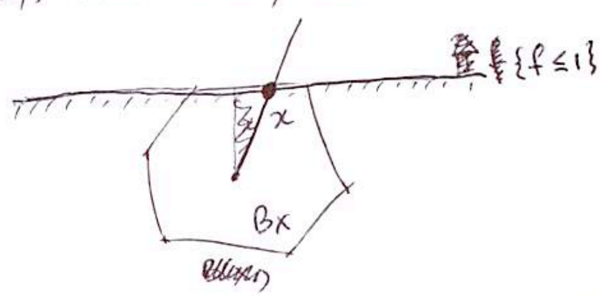
Cor (Supporting functional)

> For every $x \in X$ there exists $f \in X^*$ (called the <u>supporting functional of x</u>) such that:
>
> (i) $\|f\| = 1$;
>
> (ii) $f(x) = \|x\|$.

~~Note~~

Remark: Recall the inequality $|f(x)| \le \|f\|\,\|x\|$; (which follows from $\|f\| := \sup_{x \in X} \frac{|f(x)|}{\|x\|}$) there exists f which realizes the equality.

Corollary says that $\forall x$, there exists f which realizes the equality.



Proof On $X_0 = \mathrm{span}(x)$, define the linear functional $f_0 \in X_0^*$

by $\quad f_0(\lambda x) = \lambda \|x\|$. $\quad$ Then $\|f_0\| = 1$.

An extension $f \in X^*$ of f_0 guaranteed by Hahn-Banach Theorem clearly satisfies (i) and (ii) $\quad$ QED.

Remark Consider again the inequality

$$|f(x)| \le \|f\|\,\|x\|, \qquad x \in X, \ f \in X^*.$$

- It is <u>not</u> true that $\forall f$, $\exists x$ which realizes the equality (Ex: construct an example).

  That is, f does <u>not</u> necessarily attain its norm on some vector.

- But Cor says that every x attains its norm for some functional:

$$\boxed{\|x\| = \max_{f \in X^*} \frac{|f(x)|}{\|f\|}}.$$