Hahn–Banach Theorem

Extensions from proper closed subspaces.

Hahn–Banach Thm. Let $X_0$ be a closed subspace of a normed space $X$.

Then every $f \in X_0^*$ admits an extension $f \in X^*$ such that

$$\|f\| = \|f_0\|.$$

1) Extension by one dimension: Assume first that $X_0$ is a hyperplane in $X$,

i.e., codim $X_0 = 1$. Therefore $x \in X$.

Fix $z \in X \setminus X_0$; then every vector $x \in X$ can be uniquely represented as

$$x = az + x_0, \quad a \in \mathbb{R}, \ x_0 \in X_0.$$

Since $f_0$ is linear, $f(x) = af(z) + f(x_0) = af(z) + f_0(x_0)$.

so $f$ is also linear.

so the desired extension $f$ is determined by just one number $f(z) = c$.

W.l.o.g. $\|f_0\| = 1$, i.e., $|f_0(x_0)| \leq \|f_0\| \forall x_0 \in X_0$.

We are looking for $\|f\| = 1$, i.e., $|f(x)| \leq \|f\| \forall x \in X$.

$$f(x) \leq \|f\| \forall x \in X.$$

(4) $f(x) = c$.

For $a = 0$, (4) is true by the assumption.

For $a > 0$ and $a < 0$ ($b = -a$):

For $a > 0$, $f_0(x)$ is equivalent to $f_0(x_0/a)$.

This is equivalent to:

$$C \leq \|f_0(x_0/a) - f_0(x_0/a)\|, \quad a > 0, \ x_0 \in X_0.$$

$$C \geq \|f_0(x_1/b) - \|x_1/x_0\|, \quad b > 0, \ x_1 \in X_1.$$
Existence of such \( C \) is equivalent to
\[
\sup_{x_0, x, \in X_0} \left( \frac{1}{a} \right) \left( f_0(x_0) - \|x\|_{\frac{a}{b}} \right) \leq \inf_{x_0, x, \in X_0} \left( \left\| x \right\|_{\frac{a}{b}} - f_0(x_0) \right),
\]
\( a > 0, x_0, x \in X_0 \).

1. To the inequality
\[
\| x \|_{\frac{a}{b}} - f_0(x_0) - \| x \|_{\frac{a}{b}} - f_0(x_0) \leq \left\| 2 + \frac{x_0}{a} \right\| - f_0(x_0),
\]
\( a, b > 0, x_0, x \in X_0 \).

2. By the inequality
\[
\left\| \alpha \right\|_{\frac{a}{b}} \leq \left\| \frac{x_0}{a} + \frac{x_0}{b} \right\| + \frac{x_0}{b},
\]
This is true:
\[
x_0 \left( \frac{x_0}{a} + \frac{x_0}{b} \right) \leq \frac{b}{a} \left\| \frac{x_0}{a} + \frac{x_0}{b} \right\|. \quad \text{QED}
\]

2. Transfinite induction.

Recall Zorn's Lemma: A partially ordered set in which every chain has an upper bound contains a maximal element.

Consider the set \( \Gamma \) of pairs \((Y, g)\) where
\( Y \subseteq X \)
\( Y \) is a subspace, \( X_0 \subseteq Y \subseteq X \)
Consider the set of all extensions of \( f_0 \).

Precisely, let \( \Gamma \) be the set of pairs \((Y, g)\) where
\( Y \) is a subspace, \( X_0 \subseteq Y \subseteq X \), and \( g \in Y^X \) is an extension of \( f_0 \).

We want to show that \( \Gamma \) contains an element with \( Y = X \).

Consider the partial order on \( \Gamma \):
\[
\left( Y_1, g_1 \right) \leq \left( Y_2, g_2 \right) \text{ if } Y_1 \subseteq Y_2, \ g_2 \text{ is an extension of } g_1.
\]

Then every chain \( \left( \left( Y_x, g_x \right) \right) \subseteq \Gamma \) has an upper bound \( \left( Y, g \right) \) in \( \Gamma \)
\( Y = U Y_x, \ \ g(x) = g_x(x) \) if \( x \in Y_x \).

Hence, by Zorn's lemma, \( \exists \) a maximal element \( \left( Y, g \right) \) in \( \Gamma \).

Now, using the proof of the first part of the proof, we could extend \( g \) onto...
Consequences of Hahn-Banach Theorem.

X: normed space.

Cor (Supporting functional)

For every \( x \in X \) there exists \( f \in X^\ast \) (called the supporting functional of \( x \)) such that:

1. \( \| f \| = 1 \);
2. \( f(x) = \| x \| \).

Note:

Recall the inequality \( |f(x)| \leq \| f \| \| x \| \); (which follows from \( \| f \| = \sup_{x \neq 0} \frac{|f(x)|}{\| x \|} \))

Comment: Hahn-Banach

Corollary says that \( \forall x \), \( \exists \ f \) which realizes the equality.

Proof: On \( X_0 = \text{span}(x) \), define the linear functional \( f_0 \) on \( X_0 \)

\[ f_0(x) = \| x \| \text{.} \]

Then \( \| f_0 \| = 1 \).

An extension \( f \in X^\ast \) of \( f_0 \) guaranteed by Hahn-Banach Theorem clearly satisfies (i) and (ii) \( \Box @ D \).

Remark:

Consider again the inequality

\[ |f(x)| \leq \| f \| \| x \| \,, \quad x \in X \,, \quad f \in X^\ast \,.

It is not true that \( \forall f \), \( \exists x \) which realizes the equality (Ex: Construct an example).

That is, \( \forall f \) does not necessarily attain its norm on some vector.

But \( \forall \) Cor says that every \( x \) attains its norm for some functional:

\[ \| x \| = \max_{f \in X^*} \frac{|f(x)|}{\| f \|} \,.

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