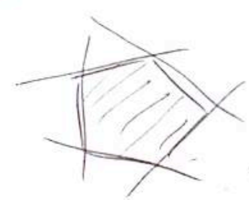
LECTURE 15

Geometric view: half-spaces ~~are~~ in X are sets of the form
$\{f \le a\}$, $a \in \mathbb{R}$, $f \in X^*$.

- Cor. says that the symmetric convex set B_X can be represented as intersection of half-spaces (which contain B_X).
- Well known for ~~polytopes~~ polyhedra in $\mathbb{R}^n$.



Cor (X^* separates points of X)

For every points $x_1 \neq x_2$ in X, there exists $f \in X^*$ such that $f(x_1) \neq f(x_2)$.

Proof Apply the previous Cor to $x := x_1 - x_2$. QED.

Second Dual

- Functionals $f \in X^*$ act on vectors $x \in X$ as $x \mapsto f(x)$.
- Vice versa, vectors $x \in X$ act on functionals $f \in X^*$ as $f \mapsto f(x)$.

In other words, vectors x may be considered as linear functionals on X^*, ~~i.e. $x \in X^{**}$ (second dual)~~. The inequality $|f(x)| \le \|x\| \|f\|$ shows that these linear functionals are continuous, i.e. $x \in X^{**}$; their norm as functionals is $\le \|x\|$. Considering the supporting functional f of x, we see that the norm of x as a functional is actually $= \|x\|$.

- Notation: $x^* \in X^*$, $\langle x^*, x \rangle = x^*(x)$.

THM (Second dual).

Let X be a normed space. Then X can be considered as a subspace of X^{**} (via the action $f \mapsto f(x)$).

Remarks

- X^{**} may be strictly lager than X,

  i.e. $c_0^* = \ell_1$, $\ell_1^* = \ell_\infty$ $\Rightarrow$ $c_0^{**} = \ell_\infty \neq c_0$

  (but $c_0 \subseteq \ell_\infty$ canonically).

- Spaces for which $X = X^{**}$ are called reflexive.

  Examples are: ℓ_p, L_p for $1 < p < \infty$.

- By Cor. on supporting functionals, every $f \in X^*$ on ~~a sp~~ a reflexive X attains its norm.

  By a result of James, this characterizes reflexive spaces.

Separation of convex sets

Remark: Not all norm axioms ~~used~~ were used in the proof of H.-B. Thm.

Just these two:

(i) $\|\lambda x\| = \lambda \|x\|$ for $\lambda \geq 0$ (positive homogeneity)

(ii) $\|x+y\| \leq \|x+y\|$ (trigle inequality).

Functions $\|\cdot\| : X \to \mathbb{R}_+^{\cup\{\infty\}}$ that satisfy (i), (ii) are called sublinear functionals.

Hahn-Banach Theorem holds for them:

if $f_0(x) \leq \|x\|$ for all $x \in X_0$

then f_0 can be extended ~~to f~~ to X while retaining (Check!)

• Sublinear functionals offer more flexibility than norms in geometric applications. They arise as Minkowski functionals of arbitrary convex sets K; ~~Similarly to the~~ (not necessarily convex as in the case of norms).

$$\|x\|_K := \inf\{t > 0 : x/t \in K\}.$$

Proposition (Minkowski functional).

Let K be a convex set in a linear vector space X, $0 \in K$.
Then $\|\cdot\|_K$ is a sublinear functional on X; ~~its level set $\{\|x\|_K = 1\} = K$~~.

~~(ii) ... Let $\|\cdot\|$ be a sublinear functional on~~ X.

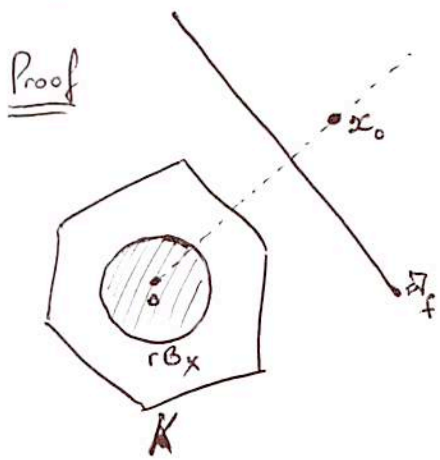
~~Then ... a~~ (See HW 09/18 #3)

Separation of convex sets.

THM (Separating a set from a point)

> Let K be an open convex subset of a normed space X; let $x_0 \in X \setminus K$. Then $\exists\, f \in X^*$, $f \neq 0$ such that
> $$f(x) \leq f(x_0) \quad \text{for all } x \in K.$$

Proof



W.L.O.G. $0 \in K$ (by translation).

~~Since A is open, it contains a ball of the form $B_X(0,r) = r\cdot B_X$ … The unit ball …~~
~~Then we can consider the Minkowski functional $\|\cdot\|_K$ of K, which by Prop~~

By Prop. p. 60, the Minkowski functional $\|\cdot\|_K$ of K is a sublinear functional.
K is open, so it contains a ball $B_X(0,r) = r \cdot B_X$ for some $r > 0$.
The inclusion of the sets implies the inequality for their Minkowski functionals:

$$r \cdot B_X \subseteq K \quad \Longrightarrow \quad \frac{1}{r}\|x\| \geq \|x\|_K, \quad x \in K.$$

Similarly to the proof of Cor. on supporting functionals (p. 57), we consider the one-dimensional subspace

$$X_0 = \operatorname{span}(x_0);$$

define a linear functional f_0 on X_0 by

$$f_0(\lambda x_0) = \lambda \|x_0\|_K.$$

~~f_0 is a bounded linear functional because~~

Then f is dominated by $\|\cdot\|$ on X_0:

$$f_0(\lambda x_0) = \|\lambda x_0\|_K \quad \text{for } \lambda \ge 0;$$

$$f_0(-\lambda x_0) = -\lambda f(x_0) = -\lambda \|x_0\|_K \le 0 \le \|\lambda x_0\|_K .$$

By Hahn-Banach Thm, f_0 can be extended to a linear functional f on X while retaining the domination:

$$f(x) \le \|x\|_K \quad , x \in X.$$

• $\underline{f \in X^*}$ (boundness):

$$f(x) \le \|x\|_K \le \frac{1}{r}\|x\| \quad , x \in X.$$

• $\underline{f(x) \le f(x_0)}$ (separation):

$\cancel{f(x) \le \|x\|_K \le f(x_0)}$ ~~because $x \in K$, $x_0 \notin K$~~

$$f(x) \le \|x\|_K \le 1 \le \|x_0\| \quad (\text{because } x \in K, x_0 \notin K)$$

$$\cancel{\le \|x_0\|} = f(x_0).$$

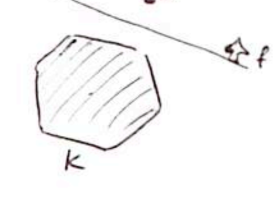
QED.

THM (Hahn-Banach, geometric form). Let A, B be disjoint convex subsets of a normed space X, and assume that A is open.
Then $\exists\, f^* \in X^*$, $f \neq 0$, $\exists\, C \in \mathbb{R}$:

$$f(a) < C \le f(b) \quad \text{for all } a \in A,\ b \in B.$$

Proof Consider $K := A - B$; then K is open and convex (check!)
Since $A \cap B = \emptyset$, we have $0 \notin K$. Using Thm p. 61, we get
$\exists f \in X^* \in X^*$, $f \neq 0$, such that

$$f(a-b) \le 0 \quad \text{for all } a \in A,\ b \in B.$$



Hence $\quad \sup_{a \in A} f(a) =: C \le \inf_{b \in B} f(b)$.

It follows that $f(a) \le C \le f(b)$. The strict inequality in the LHS follows because A is open (Check!). QED

Remark: Openness, convexity are needed