LECTURE 16

LINEAR OPERATORS

- Let X, Y be normed spaces, $T : X \to Y$ be a linear operator.
- Partial case: linear functionals ($Y = \mathbb{R}$ or $\mathbb{C}$)

Many ~~results~~ arguments ~~carry over~~ for linear functionals carry over to lin. op.

Def (Boundedness)

$T : X \to Y$ is called a bounded linear operator if $\exists\, C$ such that

$$\|Tx\| \le C\,\|x\| \quad \text{for all } x \in X.$$

The norm of T is defined as

$$\|T\| := \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{x \in S_X} \frac{\|Tx\|}{\|x\|}.$$

Remarks 1) We always have $\|Tx\| \le \|T\|\,\|x\|$,

and $\|T\|$ is the best possible const. in this inequality.

2) Compare to lin. functionals p. 48

3) $T : X \to Y$, $S : Y \to Z \Rightarrow \|TS\| \le \|T\|\,\|S\|$

Ex

Prop A linear operator $T : X \to Y$ is continuous iff T is bounded

Proof (the same as for linear functionals p.47). QED

Similarly to the dual space $X^* = \{$bounded lin. functionals on $X\}$

we consider

$$L(X,Y) = \{\text{bounded linear operators } T: X \to Y\}.$$

$(X^* = L(X,\mathbb{R}))$.

<u>Prop</u> $L(X,Y)$ is a normed space (with the operator norm)
If Y is a Banach space then $L(X,Y)$ is a Banach space.

<u>Cor</u> X^* is a Banach space (even if X is not).

<u>Proof of Prop</u> Let $T_n \in L(X,Y)$ be a Cauchy sequence:

$$\|T_n - T_m\| \to 0, \quad n,m \to \infty$$

Applying to any $x \in X$, we see that $\overset{\in Y}{T_n(x)}$ is also a Cauchy seq:

$$\|T_n(x) - T_m(x)\|$$
$$= \|(T_n - T_m)x\| \le \|T_n - T_m\| \|x\| \to 0, \quad n,m \to \infty$$

Since Y is complete, this sequence converges:

$$Tx := \lim_n T_n x.$$

- It is easily seen that T is a <u>linear</u> operator $X \to Y$. (Check!)
- <u>Boundedness</u>. $\|Tx\| = \lim_n \|T_n x\| \le \underbrace{\sup_n \|T_n\|}_{<\infty} \|x\|$ because (T_n) is Cauchy.
- <u>$T_n \to T$</u> in $L(X,Y)$: Since (T_n) is Cauchy,
  $\forall \varepsilon > 0\ \exists N$: $\|T_n - T_m\| < \varepsilon$ for $n,m > N$.
  $$\Downarrow$$
  $$\|T_n x - T_m x\| \le \varepsilon \|x\|$$
  Sending $m \to \infty$, we get $\underset{\|(T_n - T)x\|}{\|T_n x - Tx\|} \le \varepsilon \|x\|$ for $n > N$.

$\Rightarrow \|T_n - T\| \le \varepsilon$ for $n > N$. QED.

Examples of linear operators

① $X = Y = \ell_2^n$

Every $T: X \to Y$ can be represented by its matrix $(a_{ij}) = \left(\langle Te_i, e_j\rangle\right)_{i,j=1}^n =$.

$$Tx = T\left(\sum_{i=1}^n \langle x, e_i\rangle e_i\right) = \sum_{i=1}^n x_i \, Te_i$$

$$\|Tx\| \le \sum_{i=1}^n |x_i| \, \|Te_i\| \underset{CS}{\le} \left(\sum_{i=1}^n |x_i|^2\right)^{1/2} \left(\sum_{i=1}^n \|Te_i\|^2\right)^{1/2}$$

$$= \left(\sum_{i=1}^n \|Te_i\|^2\right)^{1/2} \|x\| = \left(\sum_{i,j=1}^n |a_{ij}|^2\right)^{1/2} \|x\|$$

$$= \|T\|_{HS} \|x\| \quad \text{(Hilbert-Schmidt norm)}$$

$\Rightarrow$ **Prop** Every linear operator on ℓ_2^n is bounded,

$$\|T\| \le \|T\|_{HS}$$

② $X = Y = L_2[0,1]$ let $k(t,s) \in L_2([0,1]^2)$ (kernel)

Define $T: L_2[0,1] \to L_2[0,1]$ by

$$\boxed{(Tf)(t) = \int_0^1 k(t,s) f(s)\, ds, \qquad f \in L_2[0,1]}$$

"Hilbert-Schmidt Integral operator"

$$\|Tf\|_2^2 = \int_0^1 \left| \int_0^1 k(t,s) f(s)\, ds \right|^2 dt \underset{CS}{\le} \int_0^1 dt \int_0^1 |k(t,s)|^2 ds \underbrace{\int_0^1 |f(s)|^2 ds}_{\|f\|}$$

$$\le \|k\|_2^2 \, \|f\|_2^2 .$$

$$\Rightarrow \boxed{\|T\| \le \|k\|_2}$$

Exercise: check $\|T\| = \|k\|_2$.