LECTURE 17

(3) A similar integral operator in $X=Y=C[0,1]$, ("Fredholm integral op")

$k(s,t) \in C([0,1]^2)$.

$$\|Tf\|_\infty = \max_{t\in[0,1]} \left|\int_0^1 k(t,s) f(s)\, ds\right| \le \max_t \int_0^1 |k(t,s)|\, ds \cdot \|f\|_\infty$$

$$\Rightarrow \|T\| \le \max_t \int_0^1 |k(t,s)|\, ds$$

~~Example~~

Exercise: Show ~~check~~ the equality

(4) Multiplication by a function:

~~In $C[0,1]$, consider~~ Consider $k(t) \in C[0,1]$; and the op. in $C[0,1]$.

$$(Tf)(t) = k(t) f(t).$$

Then $\|Tf\|_\infty = \max_{0\le t\le 1} |k(t) f(t)| \le \|k\|_\infty \|f\|_\infty$.

Hence $\|T\| \le \|k\|_\infty$

Exercise: prove the equality.

(5) Shift in sequence spaces

$T: \ell_p \to \ell_p$: $\quad Tx = (0, x_1, x_2, \dots)$

Then $\|T\| = 1$ (isometry)

(6) Isomorphisms, isometries

Def $T: X \to Y$ is called an ~~isometric~~ isomorphism if $T, T^{-1} \in L(X,Y)$.
~~T is onto, and $\exists c, C$: $\|Tx\| \le$~~

$\Downarrow$ $\exists C, c$:

$$c\|x\| \le \|Tx\| \le C\|x\| \quad \forall x \in X$$

~~$T: X \to Y$ is called an isometric embedding if T~~

Isometry if $C, c = 1$: $\|Tx\| = \|x\|$.

Extensions of operators (compare to functionals, lec. 13)

~~$T \in L(X,Y)$~~ $X_0 \subseteq X$ subspace; $T_0 \in L(X_0, Y)$. Then $T \in L(X,Y)$ is an extension of T if $T|_{X_0} = T_0|_{X_0}$

1. From a dense subspace

Prop (Extension by continuity) Let $X_0 \subseteq X$ be a dense subspace of a normed space X. Then every $T_0 \in L(X_0, Y)$ admits a unique extension $T \in L(X,Y)$. Moreover, $\|T\| = \|T_0\|$.

Proof: Similar to Prop p. 54 for linear functionals! QED

✓ Ex – prove!

Examples: Definition of Fourier Transform on $\mathbb{R}^{\#}$ or $\mathbb{R}^n$ – first define on Schwartz space S (of functions whose all derivatives decay faster than $\forall$ polynomials) then extend fro. S to $L_2(\mathbb{R})$ by continuity.

2. From a closed subspace.

. Not always possible, ~~as~~ Hahn-Banach Theorem does not generalize to operators.

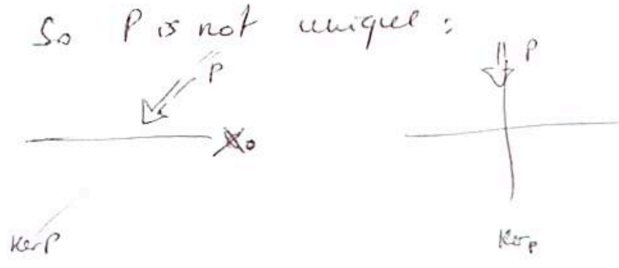
~~Either all operators can be extended from X_0 to the whole X,~~ or none. ~~Here is a characterization of such X_0.~~

Def Let $X_0 \subseteq X$ be a closed subspace of a normed space. An operator $P \in L(X, X)$ is called a *projection* onto X_0 if

(i) $P(X) \subseteq X_1$;

(ii) $Px = x$ for all $x \in X_0$, i.e. $P|_{X_0} = Id_{X_0}$

Remark 1) Example: orthogonal projections in Hilbert spaces.
But a general P does not have to be orthogonal (besides, there is no concept of orthogonality in Banach spaces).
So P is not unique:



Proposition Let X_0 be a closed subspace of a normed space X.

T.F.A.E.:

(i) There exists in X a projection onto X_0

("X is <u>complemented</u> in X")

(ii) For every normed space Y,

every $T_0 \in L(X_0, Y)$ admits an extension $T \in L(X, Y)$

$$\begin{tikzcd} X_0 \arrow[r, "T_0"] \arrow[d, phantom, "\cap"] & Y \\ X \arrow[ur, "T"'] & \end{tikzcd}$$

<u>Proof</u> (i) $\Rightarrow$ (ii) The operator $T := T_0 P$ is an extension of T.

(ii) $\Rightarrow$ (i)

$$\begin{tikzcd} X_0 \arrow[r, "Id"] \arrow[d, phantom, "\cap"] & X_0 \leq X \\ X \arrow[ur, "P"'] & \end{tikzcd}$$

~~Consider $Id: X_0 \to X_0$ the identity,~~

~~and extend it to $P: X \to X_0$~~

An extension $P: X \to X_0$ of the identity $Id: X_0 \to X_0$

(~~when considered as~~ is a projection onto X_0 (when considered as $P: X \to X$)

QED

Complementary Subspace Problem.

Not all subspaces are complemented / later we will show that $\forall$ f.d. subspace is complemented

- For example, c_0 is uncomplemented in ℓ_∞ [Phillips '40].
- Every complemented subspace of ℓ_1 is isomorphic to ℓ_1. [Pełczyński '60]
- The same fact holds if ℓ_1 is replaced by ℓ_p, $1 \leq p \leq \infty$, or c_0 [Lindenstrauss '67] (but not L_p!)
- Every Banach space that is not isomorphic to a Hilbert space has an uncomplemented subspace [Lindenstrauss – Tzafriri '71].
- There exists Banach spaces without ~~not~~ non-trivial complemented subspaces (i.e. $\dim X_0 = \infty$, $\operatorname{codim} X_0 = \infty$) [Gowers – Maurey '93]
- ~~Every sub~~ If X_0 is isomorphic to ℓ_∞ then it is complemented.

Adjoint Operators

Def Let $T \in L(X,Y)$.
$T^*: Y^* \to X^*$ is the linear operator defined as

$$(T^*f)(x) := f(Tx), \qquad x \in X,\ f \in Y^*.$$

In notation $f(x) = \langle x, f \rangle$, the ~~dual~~ adjoint operator acts as

$$\boxed{\langle x, T^*f \rangle = \langle Tx, f \rangle.}$$

Examples ① $X = Y = \mathbb{R}^n$ or $\mathbb{C}^n$.
The matrix of T^* is the transpose of the matrix of T

$$T = (t_{ij})_{i,j=1}^n \Rightarrow T^* = (\overline{t_{ji}})_{i,j=1}^n$$

② Integral operator on $L_2(0,1)$:

$$(Tf)(t) = \int_0^1 k(t,s) f(s)\, ds,$$

has adjoint T^* which is also an integral op

$$(T^*f)(t) = \int_0^1 \overline{k(s,t)}\, f(s)\, ds.$$

Proof:

$$\langle Tf, g \rangle = \int_0^1 \left(\int_0^1 k(t,s) f(s)\, ds \right) \bar{g}(t)\, dt$$

$$= \int_0^1 f(s) \left(\int_0^1 k(t,s) \bar{g}(t)\, dt \right) ds$$

$$= \int_0^1 f(s)\, \overline{\int_0^1 \overline{k(t,s)}\, g(t)\, dt}\, ds$$

$$= \langle f, T^*g \rangle$$

$$\Rightarrow (T^*g)(s) = \int_0^1 \overline{k(t,s)}\, g(t)\, dt. \quad QED$$

③ The right shift $T: \ell_p \to \ell_p$: $Tx = (0, x_1, x_2, \ldots)$
has ~~conjug~~ adjoint $T^*: \ell_q \to \ell_q$: $T^*x = (x_2, x_3, \ldots)$

Prop $T^* \in L(Y^*, X^*)$; $\|T^*\| = \|T\|$

Proof

$$\|T^*\| = \sup_{f \in S_{Y^*}} \|T^*f\|_{X^*} = \sup_{f \in S_{Y^*}} \sup_{x \in S_X} |(T^*f)(x)|$$

$$= \sup_{x \in S_X} \sup_{f \in S_{Y^*}} |f(Tx)| = \sup_{x \in S_X} \|Tx\| \quad \text{by Cor. on supporting functionals p.57 (see Remark below it)}$$

$$= \|T\|. \qquad QED$$

Further properties: 1) $T \in L(X,Y)$, $S \in L(Y,Z) \Rightarrow (ST)^* = T^*S^*$
2) $(aT + bS)^* = \bar{a}T^* + \bar{b}S^*$