LECTURE 18

FUNDAMENTAL PRINCIPLES OF FUNCTIONAL ANALYSIS

(Open Mapping Thm, Uniform Boundedness Princ, Closed Graph Thm)

~~Open Mapping Thm~~

Baire Category Thm.

- M: metric space

  $A \subseteq M$ is called <u>nowhere dense</u> if $\bar{A}$ has empty interior

  Equivalently, A is nowhere dense if, for every open ball B in M, one can find another open ball $B_1 \subseteq B$ such that $B_1 \cap A = \emptyset$

  "hole".

  <u>Remark</u>: By making B_1 slightly smaller, we can ~~get have B_1 a closed ball~~ consider closed balls.

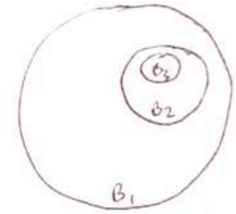
B

- A is of <u>first category</u> if $A =$ countable union of nowhere dense sets

  Otherwise, <u>second category</u>

<u>Baire Category Thm</u>. Every complete metric space is a set of second category

<u>Proof</u>: Assume M is of first category, i.e. $M = \bigcup_n A_n$, where $A_n \subseteq M$ are nowhere dense.



$\exists$ $B_1 \subseteq M$ be a closed ball, $B_1 \cap A_1 = \emptyset$

$\exists$ $B_2 \subseteq B_1$ - - - - - $B_2 \cap A_2 = \emptyset$

$\exists$ $B_3 \subseteq B_2$ - - - - - $B_3 \cap A_3 = \emptyset$.

- - - - - - -

Can make radii $\to 0$. The centers of these balls form a Cauchy sequence. By completeness, it converges to $x \in M$. Clearly, $x \in \bigcap_n B_n$. Hence $x \notin A_n$ for all n

Contradiction. QED

Open Mapping Thm

Open Mapping Thm (Banach).

Let X, Y be Banach space.

Then every onto map $T \in \mathcal{L}(X,Y)$

is an open map, i.e. ~~for every open set $B \subseteq X$,~~

~~the image $T(B)$ of every open set $B \subseteq X$ is open.~~

~~i.e.~~ T maps open sets to open sets.

Proof

Claim: it suffices to prove that $\exists \varepsilon > 0$:

$$TB_X \supseteq \varepsilon \cdot B_Y$$

Indeed, we need to show that ~~every $y \in T$~~ for every open set $U \subseteq X$, every $y \in TU$ is an interior point of TU, ~~i.e. there exists ε~~

Let $y = Tx$ for some $x \in U$. Since U is open, $\exists \delta > 0$: st

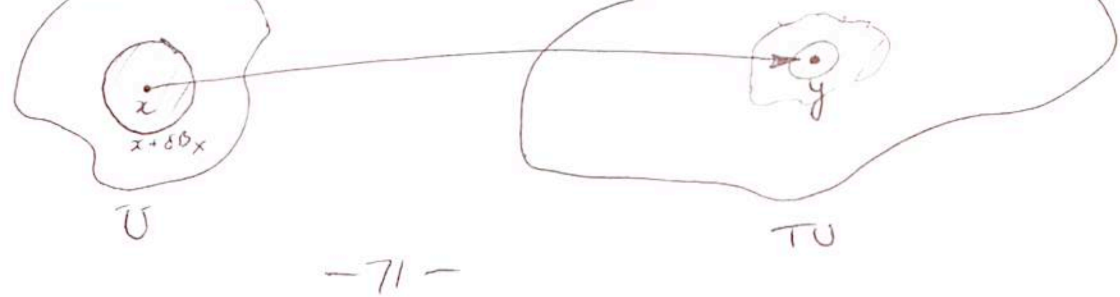
$$U \supseteq x + \delta B_X$$

Thus

$$TU \supseteq T(x + \delta B_X) = y + \delta \cdot TB_X \overset{\text{by Claim}}{\supseteq} y + \delta\varepsilon \cdot B_Y$$

~~If the Claim is true~~

hence y is an interior point of TU.



We will prove Claim using Baire Category Thm, as follows.

$$X = \bigcup_{n \in \mathbb{N}} n B_X$$

Hence

$$Y = TX = \bigcup_{n \in \mathbb{N}} n\, TB_X .$$

By Baire Category Thm, there exists $n \in \mathbb{N}$ s.t. $n \cdot TB_X$ is not nowhere dense.

By scaling, TB_X itself is not nowhere dense.

Thus $\exists\, y \in Y$, $\varepsilon > 0$ such that

$$\overline{TB_X} \supseteq y + \varepsilon B_Y .$$

By symmetry, $\overline{TB_X} \supseteq -y + \varepsilon B_Y$

Hence by convexity,

$$\boxed{\overline{TB_X} \supseteq \varepsilon B_Y .}$$

- We have almost proved Claim, except for the closure.

Generally, ~~the closure can not be removed~~ $\overline{K} \supset D \not\Rightarrow K \supset D$

but convexity ~~helps~~ of K helps:

__Lemma__ Let K be a <u>perfectly convex</u> set in $\overset{\text{a Banach space}}{Y}$, i.e. for every sequence $(x_i)_1^\infty \subseteq K$ and every $\lambda_i \geq 0$, $\sum_1^\infty \lambda_i = 1$, one has $\sum_1^\infty \lambda_i x_i \in K$.

~~Then $\overset{\circ}{\overline{K}} = \overset{\circ}{K}$~~ If $\overline{K}$ contains some open ball εB_Y then K contains $\frac{\varepsilon}{2} B_Y$.

Assume $B := \varepsilon B_Y \subseteq \overline{K}$.

Proof of lemma. ~~It suffices to show that, x~~

"$\supseteq$" is trivial: ~~It suf~~

We need to show that $x \in \mathring{\overline{K}}$ implies $x \in \mathring{K}$.

By translation we can assume that $x = 0$. So we want to show that

$$0 \in \mathring{\overline{K}} \text{ implies } 0 \in \mathring{K}.$$

There exists an open ball B ~~centered at 0~~ $= \varepsilon B_X(0)$ and such that $B \subseteq \overline{K}$.

We will show that $\frac{1}{2} B \subseteq K$, which would complete the proof.

The inclusion $B \subseteq \overline{K}$ implies that

$$B \subseteq K + \tfrac{1}{2} B$$

which in turn implies

$$B \subseteq K + \tfrac{1}{2}\left(K + \tfrac{1}{2} B\right) = K + \tfrac{1}{2} K + \tfrac{1}{4} B$$

$$\subseteq K + \tfrac{1}{2} K + \tfrac{1}{4}\left(K + \tfrac{1}{2} B\right) = K + \tfrac{1}{2} K + \tfrac{1}{4} K + \tfrac{1}{8} B$$

$$\subseteq \dots$$

$$\Rightarrow \quad B \subseteq K + \tfrac{1}{2} K + \tfrac{1}{4} K + \dots$$

$$\Rightarrow \quad \tfrac{1}{2} B \subseteq \tfrac{1}{2} K + \tfrac{1}{4} K + \tfrac{1}{8} K + \dots \subseteq K \quad \text{by perfect convexity.}$$

This completes the proof of lemma.

Exercises 1) Show that $\varepsilon B_Y \subseteq K$. 2) Show that $\mathring{\overline{K}} = \mathring{K}$.

Proof of Open Mapping Thm continued.

By the lemma, it suffices to show that $K = TB_X$ is perfectly convex.

This is indeed true: for every sequence $(Tx_i) \subset TB_X$, $x_i \in B_X$,

and $\forall \lambda_i \geq 0$, $\sum_1^\infty \lambda_i = 1$,

$$\sum_1^\infty \lambda_i T x_i = T\Big(\underbrace{\sum_1^\infty \lambda_i x_i}\Big) \subseteq TB_X$$

abs. converges. $\left\| \sum_1^\infty \lambda_i x_i \right\| \leq \sum \lambda_i \underbrace{\|x_i\|}_{\leq 1} \leq 1$.

This completes the proof. ∎