LECTURE 19

Isomorphisms & Equivalent norms

Cor (~~Open~~ Inverse Mapping Thm).

Let X, Y be Banach spaces.
Then every bijective $T \in L(X,Y)$ is an isomorphism,
i.e. $T^{-1} \in L(Y,X)$.

Proof: follows from Open Mapping Thm, which states that the preimages of open sets under T^{-1} are open $\Rightarrow$ T^{-1} is continuous.
QED

Remarks 1) Isomorphisms preserve all vital structures in normed spaces. They map open sets to open sets, closed to closed, convergent sequences to convergent sequences, complete spaces to complete spaces
(Exercise)

2) Application to solving linear systems

$$Tx = b$$

where $T \in L(X,Y)$, $b \in Y$.
Assume that solution x exists and is unique for every RHS b.
Then, by I.M.T., the solution $x = x(b)$ is continuous w.r. to b.
In other words, the solution is stable under perturbations of RHS.

3) Most ∞-dim. Banach spaces are not isomorphic.
Of all spaces $L_p[0,1]$, ℓ_q $(1 \le p, q \le \infty)$,
there are ~~only two~~ exactly two isomorphic ones: $L_2[0,1] = \ell_2$.
[A. Pelczynski] - see [Lindenstrauss - Tzafriri].

Cor

Ex

Let $T \in L(X,Y)$ be onto. Then T is a composition of a quotient map and an isomorphism:

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "q"'] & Y \\ X/\ker T \arrow[ur, "\tilde{T}"'] & \end{tikzcd}$$

(The injectivization $\tilde{T}$ considered in the HW is an isomorphism by the I.M.T).

Deduce: ~~$\forall$ lin op $X \to Y$~~ If $\dim Y < \infty$ then $\forall$ lin. op. $T: X \to Y$ is bounded

Def Two norms $\|\cdot\|$ and $|||\cdot|||$ on a linear space E are called equivalent if $\exists\, C, c > 0$ such that

$$c\|x\| \leq |||x||| \leq C\|x\| \quad \text{for all } x \in E.$$

- In other words, $\|\cdot\|$ and $|||\cdot|||$ are equivalent if the identity operator $Id: (E, \|\cdot\|) \to (E, |||\cdot|||)$ is an isomorphism.

Ex

- ~~In other wo~~ Equivalently, the two norms are equivalent iff they generate the same topology on E (Ex.).

Cor Let $\|\cdot\|$, $|||\cdot|||$ be norms on a linear space E, and E is complete with respect to both norms. If $\exists\, C$:

$$|||x||| \leq C\|x\| \quad \text{for all } x \in E$$

then the two norms are equivalent

Proof This follows from the previous Corollary applied to the identity map.

QED

- **Remark**: One can imediately deduce, for example, that $C[0,1]$ is incomplete w.r.to L_1-norm, since $\|\cdot\|_1 \leq \|\cdot\|_\infty$ but obviously the norms are not equivalent (for peaky functions, $\|f\|_1 \ll \|f\|_\infty$).

Finite dimensional Banach spaces

THM 1 Every n-dimensional normed space X is isomorphic to ℓ_2^n.

Proof Let $e_1, \dots, e_n$ be a basis of X, and consider the linear operator

$$T: \ell_2^n \to X, \qquad Tx = \sum_1^n x_i e_i \quad \text{for } x = (x_1, \dots, x_n) \in \ell_2^n.$$

To show that T is an isomorphism we need to find $M, m > 0$ s.t.

$$m \|x\|_2 \le \|Tx\|_X \le M \|x\|_2 \quad \text{for all } x \in X;$$

or, equivalently (by homogeneity),

$$m \le \|Tx\| \le M, \quad \text{for all } x \in S^{n-1} \text{ (the unit sphere of } \ell_2^n).$$

1. Upper bound:

$$\|Tx\| = \|\sum_1^n x_i e_i\| \overset{\Delta}{\le} \sum_1^n |x_i| \|e_i\| \le \underbrace{\Big(\sum_1^n |x_i|^2\Big)^{1/2}}_{= \|x\|_2} \underbrace{\Big(\sum_1^n \|e_i\|^2\Big)^{1/2}}_{= M}$$

2. Lower bound:

The function $x \mapsto \|Tx\|$ is a continuous function on a compact set S^{n-1}. By Heine-Borel Thm, it must attain its minimum; call it $m \ge 0$. If $m = 0$ then

$$Tx = \sum_1^n x_i e_i = 0 \quad \text{for some } x \in S^{n-1}$$

which contradicts the linear independence of (e_i).

Hence $m > 0$.

QED

Remark But not isometric! The infimum of $\|T\| \|T^{-1}\|$ over isomorphisms $T: X \to Y$ is called the Banach-Mazur distance $d(X, Y)$. Isometry iff $d(X, Y) = 1$.

THM [John '48]. $d(X, \ell_2^n) \le \sqrt{n}$; ~~This is sharp~~

This is sharp: $d(\ell_1^n, \ell_2^n) = \sqrt{n}$.

Cor Every two n-dimensional normed spaces are isomorphic ∎

Remark $d(X,Y) \le d(X,\ell_2^n)\, d(\ell_2^n, Y) \le \sqrt{n}\cdot\sqrt{n} = n.$

This is sharp: $\exists\, X, Y$: $d(X,Y) \ge cn$. [Gluskin '81].

Cor Every finite dimensional normed space is a Banach space

Proof $X \sim \ell_2^n$; ℓ_2^n as we know is complete.
Thus X is complete. QED

Cor Every finite dimensional subspace of a normed space is closed.

Proof Let $Y \subseteq X$ be a f.d. subspace. Then Y is complete by the previous Cor.
~~If $(y_n) \subset Y$ is a If $y_n \to y$ in Y then (y_n) is a Cauchy sequence~~ If (y_n) in Y converges to $x \in X$ then (y_n) is Cauchy in Y
$\Rightarrow$ has limit in Y. $\Rightarrow x \in Y$. QED

Cor Every linear operator on a finite dimensional B. space is bounded

Proof We know this for ℓ_2^n; $\|T\| \le \|T\|_{HS}$. By isomorphism,
this is true for all n-dim. spaces QED

EX **Remarks** This also holds for $T: X \to Y$ whenever X or Y are finite dim.

Cor Every two norms on a finite dimensional Banach space are equivalent

Proof Let $\|\cdot\|$, $|||\cdot|||$ be two norms on X. By the previous corollary,
the identity map $Id: (X, \|\cdot\|) \to (X, |||\cdot|||)$ as well as its inverse
is bounded. QED