From linear algebra,

Def (dimension), ~~The next~~

1) Let $\{e_1, \dots, e_n\}$ be a maximal set of linearly ind. vectors in E. Then n is called the dimension of E ($n = \dim E$). If no such system exists, $\dim E = \infty$.

2) The dimension is independent of a particular choice of the vectors.

3) $\forall x \in E$ can be written uniquely as a linear combin.

$$x = \sum_1^n a_i x_i \quad , \quad a_i \in \mathbb{R}$$

$\{e_1 \dots e_n\}$ is called a basis of E.

Examples: 1) F, $C(a,b)$, $L_1(a,b)$, $P(x)$ are ∞-dimensional

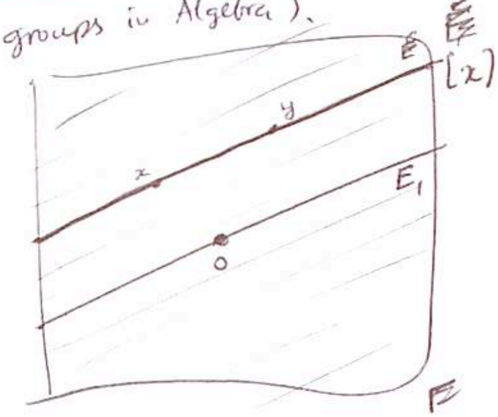
2) $\dim \mathcal{P}_n = n$; a basis is formed by the monomials $\{1, x, x^2, \dots x^n\}$.

3) $\dim \mathbb{R}^n = n$

4) all other sequence spaces discussed are ∞-dim.

LECTURE 2

Quotient spaces (compare to quotient groups in Algebra).



Let $E_1 \subset E$ subspace.

- Consider the equivalence relation on E:

$$x \sim y \overset{def}{\Leftrightarrow} x - y \in E_1$$

$E/E_1 = \{$equivalence classes (or "cosets") $[x]$ of all $x \in E\}$.

- Observe that $[x] = x + E_1 = \{x + h, \ h \in E_1\}$.

We can make E/E_1 into a linear space by defining

$$[x] + [y] := [x+y]; \qquad a(x) := [ax].$$

<u>Def</u> E/E_1 is called the <u>quotient space</u>

$\dim E/E_1$ is called the <u>codimension</u> of E_1, denoted <u>$\operatorname{codim} E_1$</u>

<u>Examples</u> 1) L_1

1) Let (Ω, Σ, μ) : measure space (think of $[a,b]$ with Lebesgue meas.)

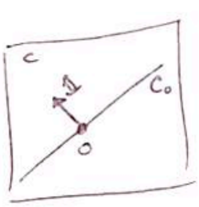
$E := \{$ all integrable functions on $\Omega \}$

$E_1 = \{$ all functions $= 0$ a.e. $\}$

Then $L_1(\Omega, \Sigma, \mu) := E/E_1$.

Thus, ~~all~~ L_1 is the space of integrable functions where we identify functions equal a.e.

2) $C_0 \subseteq C$ Then $\operatorname{codim} C_0 = 1$.



$\forall x \in c$, $x = a\mathbb{1} + z$ for some $a \in \mathbb{R}$, $z \in c_0$.

Hence $[x] = a[\mathbb{1}] + \underbrace{[z]}_{=0} = a[\mathbb{1}]$. qed.

Linear operators

Def A map $A: E \to F$ between two linear spaces E and F is ~~a linear~~ called a <u>linear operator</u> ~~(linear map)~~ ~~if~~
(transformation, map)

if
$$A(ax + by) = a\,A(x) + b\,A(y) \qquad \forall\, x, y \in E, \ \forall\, a, b \in \mathbb{R}$$
often written as $aAx + bAy$

Def
$$\ker(A) = \{x \in E : Ax = 0\}$$
$$\operatorname{Im}(A) = \{Ax : x \in E\}.$$

Exercise: A is one-to-one $\Leftrightarrow$ $\ker A = \{0\}$.

Examples
(a) $A(f) = f'$, $A: \mathcal{P}(x) \to \mathcal{P}(x)$.

(b) <u>Embedding</u> ~~operator~~: $E_1 \subseteq E$

$$A: E_1 \to E \ : \ Ax = x$$

(c) <u>Quotient</u> map: $E_1 \subseteq E$

$$A: E \to E/E_1 \ : \ Ax = [x]$$

Exercise: check linearity,
show that $\ker A = E_1$, $\operatorname{Im} A = E/E_1$ (~~onto~~) (surjective)

(d) Shift on $\forall$ sequence space: $(Af)(k) = f(k+1)$, $\forall f, \forall k \in \mathbb{Z}$

Normed spaces.

Def (normed space) Let E be a linear space.

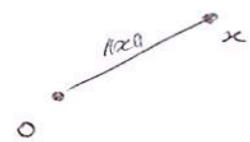
A norm $\|x\|$ for $x \in E$ is a function $E \to \mathbb{R}$ satisfying:

(i) $\|x\| \geq 0$ for all $x \in E$, $\|x\| = 0$ iff $x = 0$;

(ii) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $x \in E$, $\lambda \in \mathbb{R}$ (or $\mathbb{C}$)

(iii) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in E$.

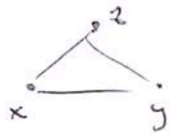
The space E equipped with a norm $\|\cdot\|$ is called a normed space, denoted $X = (E, \|\cdot\|)$.



Remarks 1) Norm $\approx$ "length" of a vector

2) Hence norm defines a metric on E;

$$d(x,y) := \|x-y\|$$



Example: check the metric axioms

$$d(x,z) \leq d(x,y) + d(y,z)$$

Δ: $\|x-z\| \leq \|x-y\| + \|y-z\|$ ok from (iii).

3) Hence ~~X is a topolog~~ a topology is defined, $\Rightarrow$ convergence

Exercise Prove that if $x_n \to x$ in X then $\|x_n\| \to \|x\|$.

Examples

1) $l_\infty = \{$all bounded sequences of real (complex) numbers$\}$

i.e. $x = (x_i)_1^\infty \in l_\infty$ iff $\sup_i |x_i| < \infty$

The norm is defined as

$$\|x\|_\infty := \sup_i |x_i|.$$

~~Exercise: check norm axi~~

Exercise: check norm axioms.

2) $c_0 = \{$all sequences converging to $0\}$,

i.e. $x = (x_i)_1^\infty \in c_0$ iff $\lim_i x_i = 0$

The norm is defined as in l_∞:

$$\|x\|_\infty := \sup_i |x_i| .$$

3) $c = \{$all convergent sequences$\}$ — the norm is also defined as in l_∞

(rarely used because c "almost" coincides with $c_0 \subseteq c$
$\operatorname{codim} c_0 = 1$.)

4) l_1: consists of all sequences $x = (x_i)_1^\infty$ satisfying

$$\|x\|_1 := \sum_{i=1}^\infty |x_i| < \infty .$$

Exercise: check norm axioms.

Note that $l_1 \subseteq c_0$ as a linear supspace. (but not with the same norm! Exercise: check that l_1 is dense in c_0)

LECTURE 3

5) More generally, for $1 \le p < \infty$,

l_p : consists of all sequences $x = (x_i)_1^\infty$ satisfying

$$\|x\|_p := \left(\sum_{i=1}^\infty |x_i|^p \right)^{1/p} < \infty .$$

Norm Axioms (i) and (ii) are straightforward;

(iii) is ↓

Minkowski Inequality: ~~For all sequences $(a_i)_1^\infty, (b_i)_1^\infty$~~ $\forall\ 1 \le p < \infty$, for any two sequences (a_i) and (b_i) (finite or infinite),

$$\boxed{\left(\sum_i |a_i + b_i|^p \right)^{1/p} \le \left(\sum_i |a_i|^p \right)^{1/p} + \left(\sum_i |b_i|^p \right)^{1/p}}$$

(Will prove later from geometric considerations).

~~Exercise Check that for every $x \in l_p$~~

Remark $l_p \subseteq l_\infty$ as a subspace (but again, not with the same norm)

Exercise For every $x \in l_p$, $\|x\|_p \to \|x\|_\infty$ as $p \to \infty$.

Explains notation $\|\cdot\|_\infty$.