From linear algebra,

**Def (dimension)**: The vector space \( E \) has **dimension** \( n \) if and only if there exists a linearly independent set of \( n \) vectors \( \{e_1, e_2, \ldots, e_n\} \) such that every vector in \( E \) can be written uniquely as a linear combination

\[ \mathbf{v} = \sum_{i=1}^{n} a_i e_i, \quad a_i \in \mathbb{R} \]

This set \( \{e_1, e_2, \ldots, e_n\} \) is called a **basis** of \( E \).

**Examples:**
1. \( \mathbb{F}_2, \mathbb{C}(a, b), \mathbb{L}(a, b), \mathbb{P}(x) \) are \( \infty \)-dimensional.
2. \( \dim \mathbb{P}_n = n; \) a basis is formed by the monomials \( \{1, x, x^2, \ldots, x^n\} \).
3. \( \dim \mathbb{R}^n = n \)
4. All other sequence spaces discussed are \( \infty \)-dim.

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**LECTURE 2**

**Quotient spaces** (compare to quotient groups in algebra).

Let \( E_1 \subset E \) subspace.

- Consider the equivalence relation on \( E \):

\[ x \sim y \iff x - y \in E_1 \]

\[ E/E_1 = \{ \text{equivalence classes } [x] \text{ of all } x \in E \} \]

- Observe that \( [x] = x + E_1 \)

\[ \frac{y}{h} = \frac{x + h}{E_1} \]
We can make \( E/E_1 \) into a linear space by defining
\[
[x] + [y] := [x + y], \quad a(x) := [ax].
\]

**Definition:**
\( E/E_1 \) is called the **quotient space**
\( \dim E/E_1 \) is called the **codimension** of \( E_1 \), denoted \( \text{codim } E_1 \).

**Examples:**
1) \( L_1 \)

Let \((\mathbb{R}, \Sigma, \mu)\) : measure space (think of \([0,1]\) with Lebesgue measure)

\[
E := \{ \text{all integrable functions on } \mathbb{R} \} \\
E_1 := \{ \text{all functions } = 0 \text{ a.e.} \}
\]

Then \( L_1 (\mathbb{R}, \Sigma, \mu) := E/E_1 \).

Thus, \( L_1 \) is the space of integrable functions
where we identify functions equal a.e.

2) \( C_0 \subseteq C \)

Then \( \text{codim } C_0 = 1 \).

\[
\forall x \in C, \quad x = a \mathbb{1} + z \quad \text{for some } a \in \mathbb{R}, z \in \mathbb{C}.
\]

Hence \( [x] = a [\mathbb{1}] + [z] = a [\mathbb{1}] \).

**q.e.d.**
**Linear operators**

**Def**
A map $A: E \to F$ between two linear spaces $E$ and $F$ is a linear called a **linear operator** (linear map) if 

Transformation map

\[ A(ax + by) = aA(x) + bA(y) \quad \forall x, y \in E, \quad a, b \in \mathbb{R} \]

often written $aAx + bAy$

**Def**

- $\ker(A) = \{ x \in E : Ax = 0 \}$
- $\text{Im}(A) = \{ Ax : x \in E \}$

Exercise: $A$ is one-to-one $\iff$ $\ker(A) = \{0\}$.

**Examples**

(a) $A(f) = f'$, $A: \mathcal{P}(x) \to \mathcal{P}(x)$.

(b) **Embedding operator**: $E \subseteq E$ 

\[ A: E \to E : Ax = x \]

(c) **Quotient map**: $E_1 \leq E$

\[ A: E \to E/E_1 : Ax = [x] \]

Exercise: check linearity,

\{ show that $\ker(A) = E_1$, $\text{Im}(A) = E/E_1$ (surjective) \}

(d) **Shift on sequence space**: $(A \cdot f)(k) = f(k+1)$, $\forall f$, $k \in \mathbb{Z}$
Normed spaces.

Def (normed space) let $E$ be a linear space.

A norm $\|x\|$ for $x \in E$ is a function $E \to \mathbb{R}$ satisfying:

(i) $\|x\| \geq 0$ for all $x \in E$, $\|x\| = 0$ iff $x = 0$;
(ii) $\|x + y\| = \|x\| + \|y\|$ for all $x, y \in E$, $\alpha \in \mathbb{R}$;
(iii) $\|\alpha x\| = |\alpha| \|x\|$ for all $x \in E$.

The space $E$ equipped with a norm $\|\cdot\|$ is called a normed space. Denote $X = (E, \|\cdot\|)$.

Remarks

1) Norm $\approx$ "length" of a vector.

2) Hence norm defines a metric on $E$:

$$d(x, y) := \|x - y\|$$

Example: check the metric axioms:

$$d(x, z) \leq d(x, y) + d(y, z)$$

$$\|x - z\| \leq \|x - y\| + \|y - z\| \quad \text{OK from (iii)}$$

3) Hence $X$ is a topology: a topology is defined, $\implies$ convergence.

Exercise: Prove that if $x_n \to x$ in $X$ then $\|x_n - x\| \to 0$.

Examples

1) $l_\infty = \{\text{all bounded sequences} \mid \text{of real (complex) numbers}\}$

i.e. $x = (x_i)$ \in $l_\infty$ iff $\sup_i |x_i| < \infty$

The norm is defined as:

$$\|x\|_{l_\infty} = \sup_i |x_i|$$

Exercise: Check norm axioms.

Exercise: Check norm axioms.
2) \( c_0 = \{ \text{all sequences converging to 0} \} \),
   i.e. \( x = (x_i), x_i \to 0 \) if \( i \lim_{i \to \infty} x_i = 0 \)
   The norm is defined as \( \|x\|_{c_0} = \sup |x_i| \).

3) \( c = \{ \text{all convergent sequences} \} \) - the norm is also defined as \( \|x\|_{c_0} \)
   (rarely used because \( c \approx \text{"almost" coincides with } c_{0,c} \) and \( c_{0,c} \) is dense in \( c_0 \).

4) \( l_1 \) consists of all sequences \( x = (x_i) \), satisfying

   \[ \|x\|_1 := \sum_{i=1}^{\infty} |x_i| < \infty. \]

   Exercise: check norm axioms.

   Note that \( l_1 \subseteq c_0 \) as a linear subspace.

   LECTURE 3

5) More generally, for \( 1 \leq p < \infty \),

   \( l_p \) consists of all sequences \( x = (x_i) \), satisfying

   \[ \|x\|_p := \left( \sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} < \infty. \]

   Note: Axioms (i) and (iii) are straightforward;

   (iii) is \( \|

   \begin{align*}
   \text{Minkowski Inequality} & : \text{For any two sequences } (a_i) \text{ and } (b_i) \text{ (finite or infinite),} \\
   & \left( \sum_{i=1}^{\infty} |a_i + b_i|^p \right)^{1/p} \leq \left( \sum_{i=1}^{\infty} |a_i|^p \right)^{1/p} + \left( \sum_{i=1}^{\infty} |b_i|^p \right)^{1/p}
   \end{align*}

   \]

   (Will prove later from geometric considerations).

   Exercise: check that for every \( x \in l_p \),

   \[ \|x\|_p \to \|x\|_{c_0} \quad \text{as } p \to \infty. \]

   Exercise: for every \( x \in l_p \), explain notation \( \|x\|_{l_\infty} \).
   
   Remark \( l_p \subseteq l_\infty \) as a subspace (but not the same norm.)