The Closed Graph Theorem

· Direct products of normed spaces

Def Let X, Y be normed spaces. Then $X \oplus Y$ is the Cartesian product

$$X \oplus Y = \{ (x,y) : x \in X, y \in Y \}$$

equipped with the norm

$$\|(x,y)\| := \|x\| + \|y\|.$$

Remarks 1. This is indeed ~~the~~ a norm (Check!)

2. If X, Y are Banach spaces then $X \oplus Y$ is a Banach space (Check!)

3. ~~An equi~~ Equivalent norms on $X \oplus Y$ are

$\sqrt{\|x\|^2 + \|y\|^2}$, $\max(\|x\|, \|y\|)$ (Check!)

Def (Graph) The graph of a linear operator $T : X \to Y$ is the following linear subspace of $X \oplus Y$:

$$\Gamma(T) := \{ (x, Tx) : x \in X \} .$$

Prop If $T \in L(X,Y)$, i.e. T is bounded then $\Gamma(A)$ is closed

Proof Let $(x_n, Tx_n) \to (x,y)$ in $X \oplus Y$.

Then $\left.\begin{array}{l} x_n \to x \implies Tx_n \to Tx \text{ by continuity} \\ Tx_n \to y \end{array}\right\} \Rightarrow y = Tx \Rightarrow (x,y) \in \Gamma(T)$

QED

THM (Closed Graph Theorem) [Banach]

Let X, Y be Banach spaces; let $T : X \to Y$ be a linear operator.

~~[illegible]~~

If $\Gamma(T)$ is closed then T is bounded, i.e. $T \in \mathcal{L}(X,Y)$

Proof $\Gamma(T)$ is a closed subspace of a Banach space $X \oplus Y$

$\Rightarrow \Gamma(T)$ is a Banach space itself.

Consider the linear operator u:

~~$u : \Gamma(T) \to X$; $u : (x, Tx) \mapsto x$~~

$$X \xrightarrow{u^{-1}} \Gamma(T) \xrightarrow{u} X$$

$$x \mapsto (x, Tx) \mapsto x$$

u is bounded, onto, and one-to-one

By the ~~[illegible]~~ Inverse Mapping Theorem, u^{-1} is bounded.

Hence $\exists\ M \geq 0$:

$$\|(x, Tx)\| \leq M \|x\|$$
$$\|$$
$$\|x\| + \|Tx\|$$

$\Rightarrow \quad \|Tx\| \leq M \|x\| \quad \Rightarrow T$ is bounded. QED

Equivalent formulation: T is bounded if and only if

$x_n \to x \in X$ and $Tx_n \to y \in Y$ imply $y = Tx$.

- Recall that boundedness $\Leftrightarrow$ continuity of T is equivalent to the following:

$$x_n \to x \in X \text{ implies } Tx_n \to Tx \in Y$$

By C.G.T, in proving boundedness of T one can ~~additionally~~ ~~as~~ automatically assume that (Tx_n) converges

Examples

① The <u>differential operator</u> $Tf = f'$, $T : C^1[0,1] \to C[0,1]$
where $C^1[0,1] = \{ f \in C[0,1] : f' \in C[0,1] \}$ is the subspace of continuously differentiable functions in $C[0,1]$.

<u>Claim</u>: <u>closed graph</u>:

$\left((\lim_n f_n)' = \lim_n (f_n') \right.$ provided f_n' converges uniformly, and $\lim_n f_n(x_0)$ exists for some x_0 $\left. \right)$

let $\left. \begin{array}{l} f_n \to f \\ f_n' \to g \end{array} \right\} \Rightarrow f_n' \to f'$ by the "Differentiation of Limit Thm"

Hence $g = f'$ $\Rightarrow$ the graph is closed.

But T is clearly <u>unbounded</u>

This is because ~~E is~~ $C^1[0,1]$ is not a Banach space with sup norm

② $Tf = f''$ treated similarly.

An application of C.G.T:

THM (<u>Hörmander</u>) $\exists\, C > 0$ such that for every $f \in C^2(0,1)$:

$$\|f'\|_\infty \le C \left(\|f\|_\infty + \|f''\|_\infty \right).$$

<u>Proof</u>. This result is equivalent to boundedness of the operator $T : (f, f'') \mapsto f'$ defined on the following ~~space~~ subspace of $C(0,1) \oplus C(0,1)$:

$$E = \{ (f, f'') : f \in C^2[0,1] \}$$

<u>E is closed</u> because E is the graph of the differential op. $f \mapsto f''$ (see Example 2 above).

Hence we can use C.G.T to prove boundedness of $T : E \to C[0,1]$. So, we want to show:

$\left. \begin{array}{l} (f_n, f_n'') \to (f, f'') \text{ in } E \\ \text{and } f_n' \to h \text{ in } C[0,1] \end{array} \right\}$ imply $f' = h$.

We have: $\left. \begin{array}{l} f_n \to f \\ f_n' \to h \end{array} \right\}$, hence $f' = h$ because the operator $f \mapsto f'$ has closed graph (see Example 1 above).

QED

One more application to self-adjoint operators: ~~[Barry]~~ (Reed-Simon III.5):

Hellinger-Toeplitz Thm.

Let T be a linear operator on a Hilbert space H.
Suppose that

$$\langle x, Ty \rangle = \langle Tx, y \rangle \quad \text{for all } x, y \in H. \qquad (*)$$

Then T is bounded, i.e. $T \in L(H,H)$

Proof It suffices to check that the graph of T is closed.
So, let $x_n \to x$, $Tx_n \to y$ in H. WTS: $y = Tx$.

Let $z \in H$; compute

$$\langle z, y \rangle = \lim_n \langle z, Tx_n \rangle$$

$$= \lim_n \langle Tz, x_n \rangle$$

$$= \lim_n \langle Tz, x \rangle$$

$$= \lim_n \langle z, Tx \rangle .$$

Since this holds for all $z \in H$, we have $y = Tx$
(by Riesz Repr. Thm) QED

Remark This thm ~~shows that~~ caused difficulties in math. physics because many operators that satisfy $(*)$ are unbounded. Hellinger-Toeplitz theorem says that such operators can not be defined everywhere on a Hilbert space.

e.g. $T = \frac{d}{dt}$ on $L_2[0,1]$

$\frac{d}{dt}(e^{int}) = \underbrace{in}_{\to \infty \text{ as } n \to \infty} \cdot e^{int}$

Ex Projections (UCD)