LECTURE 21

The Principle of Uniform Boundedness.

[Banach – Steinhaus]

THM (P.U.B.) Let X, Y be Banach spaces. (Y can be just a normed space)

Suppose a family of bounded linear operators $\mathcal{T} \subset L(X,Y)$ is pointwise bounded, i.e.

$$\sup_{T \in \mathcal{T}} \|Tx\| < \infty \quad \text{for every } x \in X.$$

Then the family $\mathcal{T}$ is uniformly bounded, i.e.

$$\sup_{T \in \mathcal{T}} \|T\| < \infty.$$

Remarks 1) $\|Tx\| \le \|T\| \cdot \|x\| \Rightarrow$ 'uniform boundedness implies pointwise boundedness.'
2) The reverse reading of P.U.B: "NOT uniformly bdd $\Rightarrow$ NOT pointwise bdd" ~~is the~~ was called by Banach & Steinhaus the "principle of condensation of singularities".

Proof Let $M(x) := \sup_{T \in \mathcal{T}} \|Tx\|$. Then $X = \bigcup_{n \in \mathbb{N}} X_n$, where

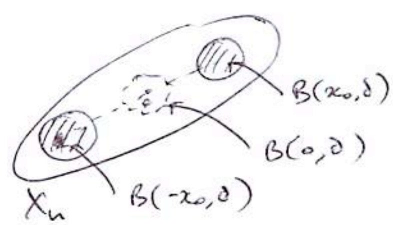
~~The level sets~~ $X_n := \{ x \in X : M(x) \le n \}$ ~~are closed (ex.)~~

are level sets.

~~and $\bigcup_{n \in \mathbb{N}} X_n = X$.~~

By Baire Category Theorem, one of X_n ~~has nonempty~~ can not be nowhere dense. Since X_n is closed (ex.), X_n has nonempty interior, i.e. contains some ball of X:

$$B(x_0, \delta) \subseteq X_n.$$



By symmetry, $B(-x_0, \delta) \subseteq X_n$

$\Rightarrow$ By convexity of X_n,

$$B(0, \delta) \subseteq X_n.$$

This means:

$$\|x\| \le \delta \text{ implies } \sup_{T \in \mathcal{T}} \|Tx\| \le n.$$

$$\Rightarrow \quad \sup_{T \in \mathcal{T}} \|Tx\| \le \frac{n}{\delta} \|x\|$$

$$\Rightarrow \quad \sup_{T \in \mathcal{T}} \|T\| \le \frac{n}{\delta}.$$

QED

Cor (Weak boundedness). Let X be a Banach space.

Suppose a subset $A \subseteq X$ is weakly bounded, i.e.

$$\sup_{x \in A} |f(x)| < \infty \quad \text{for every } f \in X^*.$$

Then A is (strongly) bounded, i.e.

$$\sup_{x \in A} \|x\| < \infty.$$

~~Suppose a subset $F \subset X^*$ is pointwise boun~~

Proof Embed $X \hookrightarrow X^{**}$ ~~isometrically~~ (identically), and consider $x \in A \subset X^{**}$ as a family of linear operators $X^* \to \mathbb{R}$.

$$\sup_{x \in A} |x(f)| < \infty \quad \text{for all } f \in X^*.$$

~~By P.U.B., $\sup_{x \in A} \|x\| < \infty$. QED~~

P.U.B. completes the proof. QED

Remark The assumption in P.U.B. can be relaxed to

$$\sup_{T \in \mathcal{J}} |f(Tx)| < \infty \quad \text{for every } x \in X,\ f \in X^*$$

by the previous Cor.

Application to convergence of Fourier series.

<u>Question</u>: when does Fourier series of a function f on $(-\pi, \pi)$ converge to f?

- in L^2 always (orthonormal basis)
- in $C[-\pi,\pi]$ – <u>not</u> always; not even pointwise:

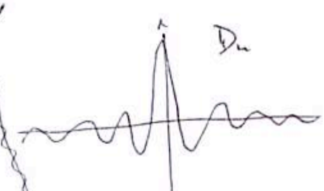
THM There exists $f \in C[-\pi, \pi]$ whose partial Fourier series

$$(S_n f)(t) = \sum_{k=-n}^{n} \hat{f}_n e^{ikt}$$

form an unbounded sequence at $t=0$.
Consequently, the Fourier series of f diverges at $t=0$.

<u>Proof</u>. Recall that $S_n f$ is the convolution of f with Dirichlet kernel:

$$(S_n f)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(s-t)\, f(s)\, ds,$$



where $$D_n(\theta) = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}.$$

$$S_n f(t) = \sum_{k=-n}^{n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) e^{-iks}\, ds \right) e^{ikt} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{\left(\sum_{k=-n}^{n} e^{-ik(s-t)} \right)}_{\substack{\text{geometric series} \\ = D_n(\theta)}} f(s)\, ds$$

~~where $D_n(\theta)=$~~

by

~~Consider the linea~~ Consider the linear functionals on $C[-\pi, \pi]$:

$$\phi_n(f) := (S_n f)(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(s) f(s)\, ds.$$

They are <u>bounded</u> lin. functionals because $D_n(s)$ is continuous $\left(\text{alternatively, because } \phi_n(f) = \sum_{k=-n}^{n} \hat{f}_n \right)$.

~~Assume that $(\phi_n(f))$ is a bounded sequen~~

• Assume that the sequence $\phi_n(f)$ is bounded for every $f \in C[0,1]$, ~~Then the it is bou~~

i.e. (ϕ_n) is a pointwise bounded sequence of lin functionals.

~~Then (ϕ_n)~~
By P.U.B, (ϕ_n) is uniformly bounded, i.e.

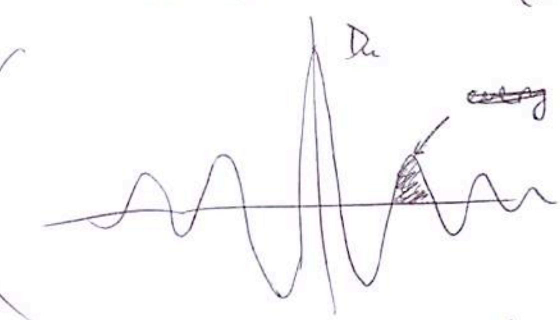
$$\sup_n \|\phi_n\| < \infty.$$

~~$\|\phi_n\| =$~~

• On the other hand, one can easily show that

$$\|\phi_n\| = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(s)|\, ds \to \infty$$

(in fact, $\sim \frac{4}{\pi^2} \log n + O(1)$)



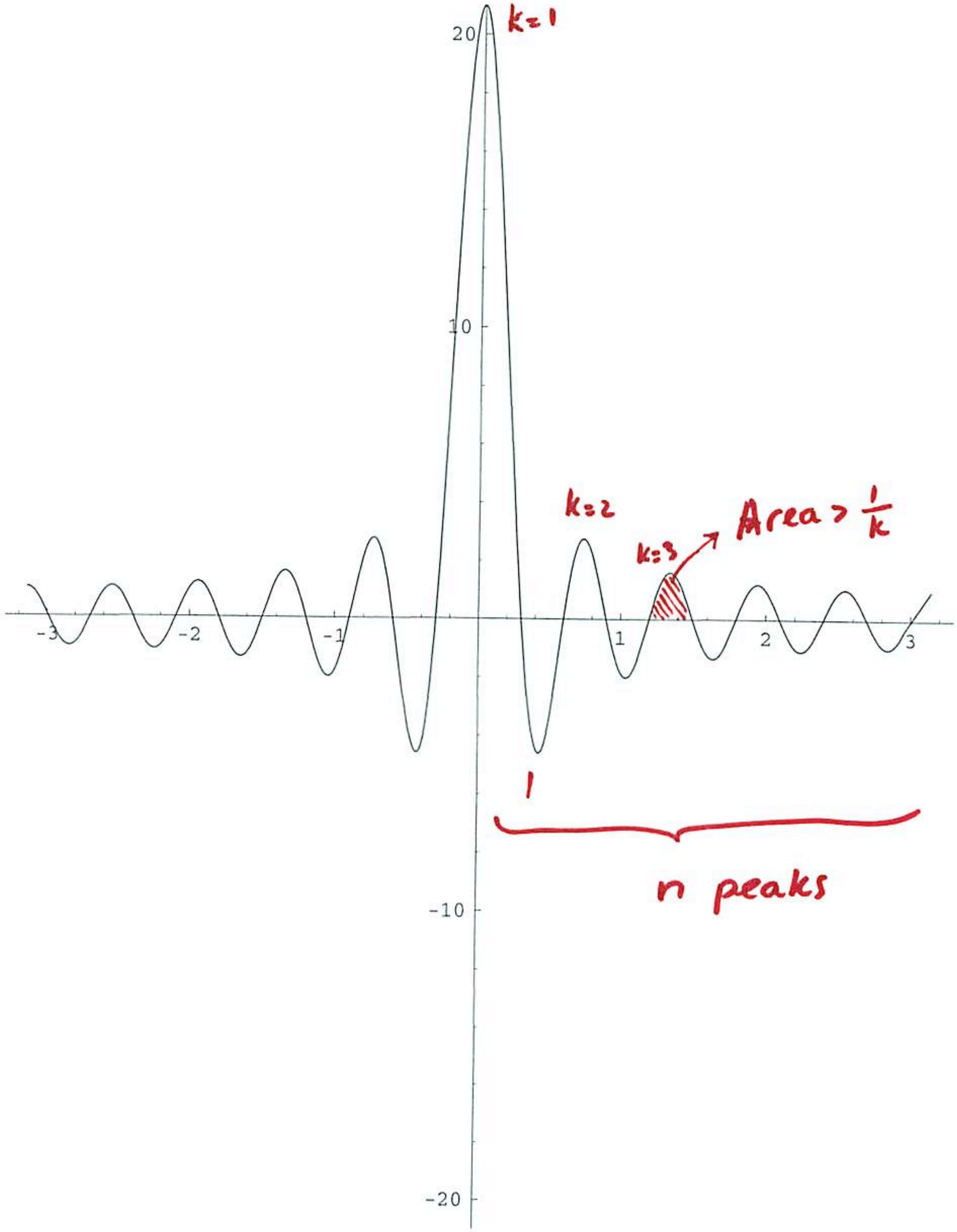
(the area under every peak is $\geq \frac{1}{k}$ hence $\|D_n\|_1 \geq \sum \frac{1}{k} \gtrsim \log n$)

This contradiction completes the proof. QED

Remarks on convergence of Fourier series:

- The proof shows that functions whose Fourier series converges form a set of first Baire category in $C[-\pi,\pi)$ ⇒ they are rare.
- If f is differentiable at x ⇒ Fourier series converges at x (Dirichlet-Dini ~~Thm~~ Condition)
- If f is continuous (or even in L^p, $p>1$) ⇒ Fourier series converges to x a.e. (Carleson '66]
- Fails for $f \in L^1$ (may diverge everywhere) (Kolmogorov).

 Dirichlet Kernel, $n=10$



Dirichlet Kernel, $n = 50$