Compact sets in Banach spaces

1. Compactness is a substitute for finite dimensionality.

2. Recall the notion of compactness.

In general, topological spaces $X$:

Def: $A \subseteq X$ is compact if every open cover of $A$ contains a finite subcover,

$$A \subseteq \bigcup_{i=1}^{n} U_{i}, \text{open} \implies \exists \alpha_{1}, \ldots, \alpha_{n} \text{ s.t. } A \subseteq \bigcup_{i=1}^{n} U_{\alpha_{i}}.$$ 

Properties:

- Compact sets are closed.
- Closed subsets of compacts are compacts.
- Continuous images of compact sets are compact sets.
- Continuous functions on compact sets are uniformly continuous, and they attain max, min.

Def: $A \subseteq X$ is precompact if $\overline{A}$ is compact.

Compactness:

(2) In metric spaces $X$

THM: For $A \subseteq X$, T.F.A.E:

(i) $A$ is precompact;
(ii) Every sequence $x_{n} \in A$ has a Cauchy subsequence (hence converges if $X$ is complete);
(iii) For every $\varepsilon > 0$, there exist an $\varepsilon$-net of $A$ (a set $N_{\varepsilon}$ such that $A \subseteq \bigcup_{\alpha \in N_{\varepsilon}}$, $d(x, y) < \varepsilon$).

As a consequence, compact sets are always closed and bounded.
Compactness in finite dimensional spaces

**THM (Heine-Borel)**

\[ A \subseteq \mathbb{R}^n \text{ is compact iff } A \text{ is closed and bounded.} \]

Since all Banach spaces \(\mathbb{R}^n\)-dimensional normed spaces are homeomorphic to \(\mathbb{R}^n\) (more accurately, to \(l_2^n\) in Heine-Borel theorem), we have this general result:

**Cor.** A subset \(A\) of a normed space \(X\) is compact iff \(A\) is closed and bounded.

Compactness in infinite dimensional spaces

The result above fails in \(\infty\)-dimension spaces.

For example, the canonical \(l_2^n\) orthonormal basis in \(l_2\) is not compact, as it does not contain a convergent subsequence \((\frac{1}{\sqrt{2}} \leq e_i, \|e_i\| = \frac{1}{\sqrt{2}})\). 

**Approximation by f.d. subspaces**

Because let \(A\) be a bounded set, a precompact set in a normed space \(X\).

Then, for every \(\varepsilon > 0\), there exists a finite dimensional subspace \(Y \subseteq X\) which forms an \(\varepsilon\)-net of \(A\).

**Proof:** Let \(N_\varepsilon\) be a finite \(\varepsilon\)-net of \(A\); \(Y := \text{span}(N_\varepsilon)\) does the job \(\square\).

Ex: prove a converse statement.
Theorem (F. Riesz). The unit ball of an infinite-dimensional normed space $X$ is never compact.

Proof. Suppose $B_X$ is compact. By lemma (p. 89), we can find a finite-dimensional subspace $Y \subseteq X$ which forms an $\frac{1}{2}$-net of $B_X$, i.e., $\text{dist}(x, Y) \leq \frac{1}{2}$ for all $x \in B_X$.

Since $X/Y$ is a nonzero Banach space, we can find a vector $\tilde{x} \in X/Y$, $\|\tilde{x}\| = 0.9$, and its representative $x \in x + Y$ such that $\|x\| \leq 1$.

Hence $x \in B_X$ and $\text{dist}(x, Y) = 0.9$ and $x \in B_X$.

The contradiction completes the proof.

Compactness in concrete Banach spaces (survey)

Theorem (Compactness in $c_0$). A set $A \subseteq c_0$ is precompact if there exists $x \in c_0$ and vector $x \in c_0$ which pointwise dominates all vectors of $A$.

For all $a \in A$ and $i = 1, 2, \ldots$. 

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THM (convergence on compacta)

Let $X, Y$ be Banach spaces. Assume that for some $T_n, T \in \mathcal{L}(X, Y)$,

$T_n \to T$ pointwise (i.e. $T_n x \to Tx$ for all $x \in X$).

Then on every precompact set $A \subseteq X$, $T_n$ converges to $T$ uniformly (i.e. $\|T_n x - Tx\| \leq \varepsilon_n \to 0$ for all $x \in A$).

Proof:

$(T_n)$ is pointwise convergent $\Rightarrow$
pointwise bounded $\Rightarrow$ bounded, i.e. $\|T_n\| \leq C$ $\forall n \in \mathbb{N}$.

Let $\varepsilon > 0$, let $N$ be an $\varepsilon$-net of $A$.

We have $T_n x \to Tx$ for every $x \in N$.

$T_n \to T$ pointwise on $N$. By fullness of $N$,

$\Rightarrow$ $T_n \to T$ uniformly on $N$: $\forall n \in \mathbb{N}$ $\forall y \in N$: $\|T_n y - Ty\| \leq \varepsilon$ for all $y \in N$.

For every $x \in A$, $\exists y \in N$: $\|x - y\| \leq \varepsilon$ $\Rightarrow$

$\|T_n x - Tx\| = \|(T_n - T)x\| \leq \|T_n - T\| \|x\| + \|T_n - T\| \|x - y\|$

$\leq \varepsilon C + (\|T_n\| + \|T\|) \|x - y\|$

$\leq \varepsilon C + 2 \varepsilon C$.

We proved: $\forall \varepsilon \in \mathbb{R} \forall n \in \mathbb{N}$ $\forall x \in X$: $\|T_n x - Tx\| \leq (2C) \varepsilon$. Q.E.D.
Compactness in concrete normed Banach spaces (survey)

**THM (C[a,b] : Arzela)**  
A ≤ C[a,b] is precompact iff:

1. A is (uniformly) bounded;
2. A is equicontinuous: \( \forall \varepsilon > 0 \exists \delta > 0 : \|f(t) - f(t')\| < \varepsilon \) for all \( f \in A \) and \( \|t - t'\| < \delta \).

Example: the set of all absolutely continuous functions.

**THM (l_p) \( 1 \leq p \leq \infty \)**  
A ≤ l_p is precompact iff:

1. A is bounded;
2. A has uniformly decaying tails: \( \forall \varepsilon > 0 \exists N \in \mathbb{N} : \sum_{n=N}^{\infty} |x_n| < \varepsilon \) for all \( x \in A \).

Example: Hilbert space; \( \{x \in l_2 : \|x\|_2 = 1, x_{2k+1} \leq \frac{1}{2^k}, x_{2k} \leq \frac{1}{2^k}, \ldots \} \).

**THM (\ell_1)**  
A ≤ \ell_1 \( [0,1] \) is precompact iff:

1. A is bounded;
2. A is "uniformly continuous on average": \( \forall \varepsilon > 0 \exists \delta > 0 \) such that \( \int |f(t+\delta) - f(t)| \, dt < \varepsilon \).