Compact sets in Banach spaces

- Compactness is a substitute for finite dimensionality.

Recall the notion of compactness

(1) Compactness In general topological spaces X:

Def $A \subseteq X$ is *compact* if every open cover of A contains a finite subcover,
(i.e. $A \subseteq \bigcup_\alpha U_\alpha$, U_α open $\Rightarrow \exists\, \alpha_1, \dots \alpha_n$: $A \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$.)

Properties
- Compact sets are closed
- Closed subsets of compacts are compacts
- Continuous images of compact sets are compact sets.
- Continuous functions on compact sets are uniformly continuous, and they attain max, min.

Def $A \subseteq X$ is *precompact* if $\bar{A}$ is compact.

(2) Compactness In metric spaces X

THM For $A \subseteq X$, T.F.A.E:

(i) A is precompact;

(ii) Every sequence $x_n \in A$ has a Cauchy subsequence (hence converges if X is complete);

(iii) For every $\varepsilon > 0$, there exists an ε-net of A (a set $N_\varepsilon \subseteq A$ such that $\forall x \in A\ \exists y \in N_\varepsilon$ s.t. $d(x,y) \leq \varepsilon$)

- As a consequence, compact sets are always closed and bounded.

Compactness in finite dimensional spaces

THM (Heine-Borel).

$A \subseteq \mathbb{R}^n$ is compact iff A is closed and bounded.

Since all ~~Banach spaces~~ n-dimensional normed spaces are isomorphic to $\mathbb{R}^n$ (more accurately, to ℓ_2^n in Heine-Borel thm), we have this general result:

~~COR A subset A of a normed space~~ fin

COR Let X be a finite dimensional normed space. $A \subseteq X$ is compact iff A is closed and bounded.

Compactness in infinite dimensional spaces

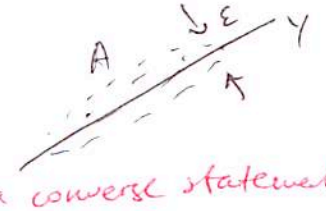
The result above fails in ∞-dim. spaces.

For example, the canonical orthonormal basis (e_i) in ℓ_2 is not compact, as it does not contain a convergent subsequence ($\|e_i - e_j\| = \frac{1}{\sqrt{2}} \not\to 0$)

Lemma (Approximation by f.d. subspaces)

Let A be ~~a bounded set~~ a precompact set in a normed space X. Then, for every $\varepsilon > 0$, there exists a finite dimensional subspace $Y \subseteq X$ which forms an ε-net of A

Proof Let N_ε be a finite ε-net of A; $Y := \mathrm{span}(N_\varepsilon)$ does the job. QED

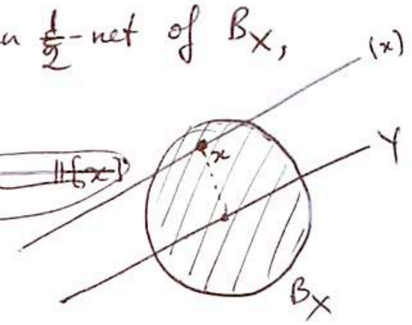


Ex: prove a converse statement.

THM (F. Riesz). The unit ball of an infinite-dimensional normed space X is never compact.

Proof Suppose B_X is compact. By lemma (p. 89), we can find a finite dimensional subspace $Y \subseteq X$ which forms an $\frac{1}{2}$-net of B_X,

i.e. $\mathrm{dist}(x, Y) \le \frac{1}{2}$ for all $x \in B_X$.



~~Then, This means that for every $[x] \in X/Y$, $\|[x]\|$~~
~~$\Rightarrow$ In X/Y, $\|[x]\| \le \frac{1}{2}$~~

Since X/Y is a nonzero space, we can find a coset

$[x] \in X/Y$, $\|[x]\| = 0.9$,

and its representative $x \in [x]$ such that

$$\|x\| \le 1.$$

Hence ~~$x \in B_X$ and di~~ $\mathrm{dist}(x, Y) = 0.9$ and $x \in B_X$.

The contradiction completes the proof. QED

~~Compactness in concrete Banach spaces (Survey)~~

~~**THM** (Compactness in c_0) $A \subseteq c_0$ is a precompact~~

~~A set $A \subseteq c_0$ is precompact iff there exists a vector $x \in c_0$ which pointwise dominates all vectors in A~~

~~$|a_i| \le |x_i|$ for all $a \in A$ and $i = 1, 2, \dots$~~

THM (convergence on compacta)

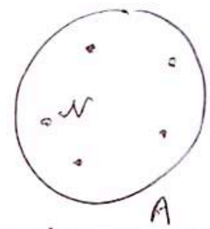
Let X, Y be Banach spaces. Assume that for some $T_n, T \in \mathcal{L}(X,Y)$,

$$T_n \to T \ \text{pointwise} \qquad (\text{i.e. } T_n x \to Tx \ \text{for all } x \in X).$$

Then on every ~~compa~~ precompact set $A \subseteq X$, T_n converges to T uniformly (i.e. $\|T_n x - Tx\| \le \varepsilon_n \to 0$ for all $x \in A$)

Proof (T_n) is pointwise convergent $\Rightarrow$ pointwise bounded $\underset{\text{P.U.B}}{\Longrightarrow}$ bounded, i.e. $\|T_n\| \le C \quad \forall n \in \mathbb{N}$.

Let $\varepsilon > 0$, let $\mathcal{N}$ be a finite ε-net of A.



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~~We have $T_n x \to Tx$ for every $x \in \mathcal{N}$.~~

$T_n \to T$ pointwise on $\mathcal{N}$. By finiteness of $\mathcal{N}$,

$\Rightarrow$ $T_n \to T$ uniformly on $\mathcal{N}$:

$$\exists N \ \forall n > N : \ \|T_n y - Ty\| \le \varepsilon \ \text{ for all } y \in \mathcal{N}.$$

For every $x \in A$, $\exists y \in \mathcal{N}: \ \|x - y\| \le \varepsilon \ \Rightarrow$

$$\|T_n x - Tx\| = \|(T_n - T)x\| \le \|(T_n - T)y\| + \|(T_n - T)(x-y)\|$$

$$\le \varepsilon + (\|T_n\| + \|T\|)\,\|x - y\|$$

$$\le \varepsilon + 2C\varepsilon.$$

We proved: $\forall \varepsilon \ \exists N \ \forall n \ge N : \ \|T_n x - Tx\| \le (1+2C)\varepsilon$. QED.

Compactness in concrete ~~normed~~ Banach spaces (survey).

THM (in $C(a,b)$: Arzelà-Ascoli) $A \subseteq C[a,b]$ is precompact iff:

(i) A is (uniformly) bounded;

(ii) A is equicontinuous: ($\forall \varepsilon > 0 \; \exists \delta > 0$: $|s-t| < \delta$ implies $|f(s)-f(t)| \leq \varepsilon$ for all $f \in A$.)

Example: the set of all 1-Lipschitz functions

THM (in ℓ_p, $1 \leq p < \infty$). $A \subseteq \ell_p$ is precompact iff:

(i) A is bounded;

(ii) A has uniformly decaying tails:

$\forall \varepsilon > 0 \; \exists N$ s.t. $\sum_{i > N} |x_i|^p \leq \varepsilon$ for all $x \in A$

Example: Hilbert cube: $\{x \in \ell_2^n : |x_1| \leq 1, \; |x_2| \leq \frac{1}{2}, \; |x_3| \leq \frac{1}{3}, \ldots\}$

THM (in L_1): $A \subseteq L_1[0,1]$ is precompact iff:

(i) A is bounded;

(ii) A is "uniformly continuous on average":

$\forall \varepsilon > 0 \; \exists \delta > 0$ such that ~~for any τ~~

$|\tau| < \delta$ implies $\int_0^1 |f(t+\tau) - f(t)| \, dt < \varepsilon$.