LECTURE 23

Weak convergence

~~Weak, weak* topologies~~

Def A sequence (x_n) in a Banach space X <u>converges weakly</u> to $x \in X$ denoted by $x_n \xrightarrow{w} x$, if
$$f(x_n) \to f(x)$$
for every $f \in X^*$

~~Remark • Strong convergence clearly implies weak convergence, • But not vice versa.~~

• Strong convergence clearly implies weak convergence, but not vice versa:

EXAMPLE An orthonormal sequence in a Hilbert space converges weakly to zero.

Proof : $(e_n) \subset H$ orthonormal. Suffices to show:
$$\langle e_n, x\rangle \to 0 \quad \text{for every } x \in X.$$
This follows from Bessel's inequality
$$\sum_1^\infty |\langle e_n, x\rangle|^2 \le \|x\|^2.$$
QED

(Compare to HW).

PROP Weakly convergent sequences are bounded.

Proof Follows from P.U.B. (Cor. p. 83): weakly convergence $\Rightarrow$ weak boundedness $\Rightarrow$ strong boundedness
QED

Cor If $x_n \xrightarrow{w} x$ then $\|x\| \le \liminf \|x_n\|$

Proof Choose $f \in X^*$ so that $\|f\| = 1$, $|f(x)| = 1$. Then $f(x_n) \le \|x_n\|$
Taking liminf of both sides, we complete the proof.

◦ Consider making example on w: convergence of measures in probability theory.

THM (Criterion of weak convergence)

Let $A \subseteq X^*$ be a dense set; $x_n, x \in X$. TFAE:

(i) $x_n \xrightarrow{w} x$

(ii) (x_n) is bounded and $f(x_n) \to f(x)$ for all $f \in A$.

Proof (i) $\Rightarrow$ (ii) follows by Prop. p. 93

(ii) $\Rightarrow$ (i): Let $g \in X^*$, let $\varepsilon > 0$. Choose $f \in X^*$ such that $\|g-f\| \le \varepsilon$.

Then

$$\cancel{\limsup |g(x_n) - g(x)| =}$$

$$\limsup |g(x_n - x)| \le \underbrace{\limsup |f(x_n - x)|}_{=0} + \limsup |(g-f)(x_n - x)|$$

$$\le \|g-f\| \underbrace{\limsup \|x_n - x\|}_{\le C} \le C\varepsilon.$$

QED

A more general form of Example p. 93: in sequence spaces X, coord. functionals $\{e_n^*\}$ span a dense ~~subset~~ subspace of X^* $\Rightarrow$

~~THM (Weak convergence in c_0, ℓ_p, $1<p<\infty$).~~
~~Let (x_n) be a bounded sequence in $X = c_0$ or ℓ_p, $1 < p < \infty$.~~
~~Then x_n converges to x weakly if x_n converges to~~

THM (Weak convergence in c_0, ℓ_p, $1<p<\infty$).

Let (x_n) be a sequence in $X = c_0$ or ℓ_p, $1<p<\infty$.
Then x_n converges weakly to $x \in X$ iff x_n is bounded and converges pointwise to x,
i.e. $x_n(i) \to x(i)$ for all $i \in N$.

QED

~~THM Weak convergence in ℓ_1 is equivalent to (strong) convergence (Schur property)~~
~~Remark By THM, Schur property holds for every every finite dim. B. space has Schur property ($X \cong \ell_2^n$)~~

Remarks 1) The same criterion holds in $C(K)$, K compact.
(by Lebesgue dominated convergence thm).

2) The criterion fails for ℓ_1.

$\boxed{\text{Weak convergence in } \ell_1 \text{ is equivalent to strong convergence}}$

(Schur property).

3) Every finite dimensional Banach space ~~also~~ has Schur property
(~~as~~ X is isomorphic to ℓ_2^n; ~~thus~~ for ℓ_2^n the result follows from Thm p.94).

~~A note~~
~~THM (Weak convergence in~~

THM (Weak convergence in L_p)

One more example: $\mathbb{1}_{[n,n+1]} \xrightarrow{w} 0$ in $L_p(\mathbb{R})$, $1<p<\infty$.

$$\left(\text{For every } g \in L_q(\mathbb{R}) = L_p(\mathbb{R})^*, \quad |\langle \mathbb{1}_{[n,n+1]}, g\rangle| = \left| \int_n^{n+1} g(t)\,dt \right| \le \int_n^{n+1} |g(t)|\,dt \le \left(\int_n^{n+1} |g(t)|^q\,dt \right)^{1/q} \text{ by Hölder} \right)$$

$\to 0$ as $\int_{-\infty}^{\infty} |g(t)|^q dt < \infty$.

More generally,

THM (Weak convergence in L_p).

Let (f_n) be a sequence in $L_p(\mathbb{R})$, $1<p<\infty$.
Then f_n converges weakly to $f \in L_p(\mathbb{R})$ iff (f_n) is bounded and

$$\int_E f_n \to 0 \text{ for every } E \subset \mathbb{R},\ |E|<\infty.$$

~~(Can be proved by~~ choosing $g \in L_q(\mathbb{R})$ and approximating g

Proof: A similar use of Thm p. 94. Step functions of compact support are dense in $L_q(\mathbb{R})$. Each such function is a finite sum of $\mathbb{1}_E$, $|E|<\infty$.

QED.