Lemma (Mazur) Let $x_n \xrightarrow{w} x$ in X.

Then $x \in \overline{\mathrm{conv}}\,(x_n)$.

Exercise: $\{x\} = \bigcap \overline{\mathrm{conv}}\,(x_i)_{i \geq n}$

Proof Suppose $x \notin K := \overline{\mathrm{conv}}\,(x_n)$ (closed convex set); then x can be strictly separated from K (see p. 62):

there exists $f \in X^*$, such that

$$\sup_{x \in K} f(x) < f(x)$$

Since $x_n \in K$, we have

$$\sup_n f(x_n) < f(x).$$

This contradicts weak convergence. QED

Exercise: Consider the seq. $x_n = (\underbrace{1,1,\dots,1}_{n},0,\dots)$ in l_∞. Show that ~~$x_n \to (1,1,\dots)$ comp~~ x_n does not weakly converge by Mazur's lemma ~~Hence Deduce that the criterion on~~ ~~Using Mazur's lem~~. Deduce that the criterion for ℓ_p (Thm p. 94) does not hold for l_∞.

Weak topology — LECTURE 24

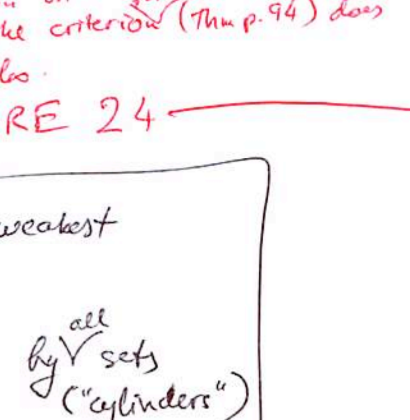
Def • The weak topology on a Banach space X is the weakest topology in which all maps $f \in X^*$ are continuous.

• Equivalently, the basis of w-topology is given by all sets ("cylinders") of the form

$$\{x \in X : |f_i(x - x_0)| < \varepsilon,\ i = 1, \dots, N\},$$

where $x_0 \in X$; $f_1, \dots, f_N \in X^*$, $\varepsilon > 0$

Remarks 1) The basis sets are clearly open in the (strong) norm topology of X. Hence w-topology is weaker than strong topology.



2) The basis sets are rather large as they contain subspaces of finite codimension $\{x \in X : f_i(x) = 0,\ i = 1 \dots N\}$.

In particular, every w-open set is unbounded.

~~• The notion of w-convergence defined in the previous lecture is clearly consistent with this def. of w-topology~~

~~• We can now talk about w-convergence (already defined) and consistent with the definition topological definition, w-closedness, w-boundedness, w-compactness, etc.~~

• The convergence in this topology is clearly the w-convergence defined in the previous lecture. But we can now broaden the picture, and talk about w-closedness, w-compactness, etc.

• In particular, P.U.B. (Cor. p. 83) ~~implies~~ can be read as:

PROP Weak boundedness is equivalent to strong boundedness.

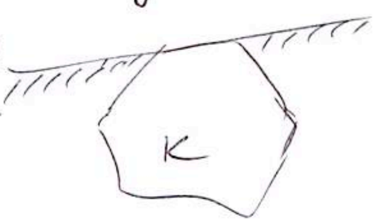
$\sup_{a \in A} |f(x)| < \infty$ for all $f \in X^*$

PROP Let $K \subseteq X$ be a convex set. Then K is w-closed iff K is closed.

Proof $(\Rightarrow)$ trivial.

$(\Leftarrow)$ By a consequence of Hahn-Banach theorem, the closed and convex set K is the intersection of the half-spaces containing K.

$$\text{half-spaces} = \{x \in X : f(x) \le a\} \text{ for some } f \in X^*, a \in \mathbb{R}$$



Each such subspace is clearly w-closed.
Hence K = intersection of w-closed sets $\Rightarrow$ K is w-closed. QED.

• Remark For non-convex sets, this is usually false. (in ∞-dim. spaces)
~~Show, for example, that the set ... which is clearly closed, is not w-closed.~~
For example, the complement of any bounded set in X is closed but not w-closed.

Weak* convergence ~~topology~~

On X^*, in addition to w-topology there is another natural weak topology (when one considers $X \subset X^{**}$).

Def (w^*-convergence) A sequence (f_n) in X^* converges weakly to $f \in X^*$ denoted by $f_n \xrightarrow{w^*} f$ if

$$f_n(x) \to f(x) \quad \text{for every } x \in X.$$

- In other words, w^* convergence can be tested on a subset X of X^{**}. Hence ~~w-top~~

~~w-topology implies w^*~~

w-convergence implies w^*-convergence in X^*.

- In reflexive spaces $(X = X^{**})$ w-convergence = w^* convergence.

In non-reflexive spaces, there are useful examples:

Example (Weak convergence of measures).

Recall: A sequence of regular Borel measures μ_n on $\mathbb{R}$ converges weakly to a Borel measure μ if

$$\int f \, d\mu_n \to \int f \, d\mu \quad \text{for every } f \in C(\mathbb{R}).$$

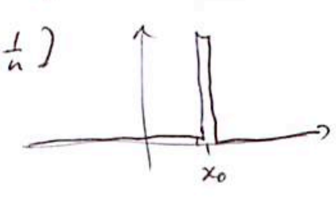
~~In particular~~ In probability theory, when μ_n, μ are probability measures, this ~~equiv~~ can be written as $\mathbb{E}_n f \to \mathbb{E} f$ ($\mathbb{E}_n, \mathbb{E}$ w.r.to μ_n, μ)

If supports of μ_n, μ are compact, (say in $[a,b]$), then the definition is equivalent to

$$\mu_n \xrightarrow{w^*} \mu \quad \text{in } (C[a,b])^*$$

So, more correctly one should call this w^* convergence of measures.

Example: (Delta function): uniform measures on $[x_0 - \frac{1}{n}, x_0 + \frac{1}{n}]$ weakly converge to δ_{x_0}



Ex

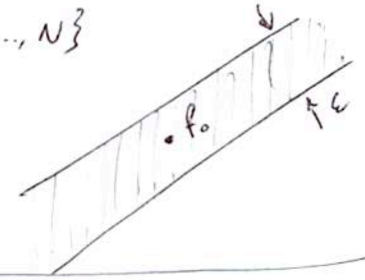
Weak* topology. Alaoglu Thm

Def • The <u>weak* topology</u> on X^* is the weakest topology in which the maps $f \mapsto f(x)$ from X^* to $\mathbb{R}$ is continuous for every $x \in X$.

• Equivalently, the <u>basis</u> of w-topology is given by all sets ("cylinders") of the form

$$\{ f \in X^* : |(f - f_0)(x_i)| < \varepsilon, \ i = 1, \dots, N \}$$

where $f_0 \in X^*$; $x_1, \dots, x_N \in X$, $\varepsilon > 0$.



Remark By the first part of the definition (and thinking of $x \in X \subset X^{**}$), we see that <u>w^*-topology is weaker than w-topology</u>.

(but obviously the same on reflexive spaces).

THM (<u>Alaoglu</u>) For every Banach space X, the closed unit ball B_{X^*} is w^*-compact

Remark But not compact as we know! Alaoglu thm thus allows for compactness arguments to hold even in ∞-dim. Banach spaces.

Proof Quickly follows from Tychonoff's theorem on products.

To this end, we seek to embed B_{X^*} into the product space of intervals

$$K := \prod_{x \in X} [-\|x\|, \|x\|] = \{ f : X \to \mathbb{R} \text{ s.t. } |f(x)| \le \|x\| \text{ for all } x \in X \}$$

equipped with product ("Tychonoff's") topology (base sets $= \prod_x A_x$ where only finitely many of A_x are $\ne \mathbb{R}$). (A_x open in $\mathbb{R}$)

• By Tychonoff's theorem, K is compact. ~~This is the weakest topology to ..~~

• We can identify $f \in B_{X^*}$ with $(f(x))_{x \in B(X)} \in K$

With this identification, w^*-topology on B_{X^*} = product topology on K (think of base sets).

• It remains to show that B_{X^*} is w^*-closed in K then

This is the weakest topology in which all point evaluation functions $f \mapsto f(x)$ are continuous for all $x \in X$.
$K \to \mathbb{R}$

- By Tychonoff's theorem, K is compact.

- We can identify $f \in B_{X^*}$ with $(f(x))_{x \in X} \in K$.

With this identification, w^*-topology on B_{X^*} = product topology on K (think of base sets)

~~Hence this iden~~ In other words, this identification is a homeomorphic embedding of B_{X^*} into K.

- To complete the proof, it remains to check that B_{X^*} is (w^*)closed in K.

~~Observe that $K = \{$all functions $f: X \to \mathbb{R}$ s.t. $|f(x)| \le \|x\|$ for all $x \in X\}$~~

and B_{X^*} are all linear functions in K. Hence

$$B_{X^*} = \bigcap_{\substack{x,y \in X \\ a,b \in \mathbb{R}}} \underbrace{\{f \in K : f(ax+by) = af(x) + bf(y)\}}_{= B_{x,y,a,b}}$$

It remains to check that $B_{x,y,a,b}$ is (w^*)-closed in K.

Note that $B_{x,y,a,b}$ is the preimage of the closed set $\{0\}$ under the map $f \mapsto f(ax+by) - af(x) - bf(y)$, which is continuous on K (recall that point evaluation functions are continuous on K).

Hence $B_{x,y,a,b}$ is closed in $K \Rightarrow B_{X^*}$ is closed in K.

QED