LECTURE 25

Krein - Milman Theorem

• Recall: the convex hull of a subset K of a vector space is

$$\mathrm{conv}(K) = \text{minimal convex set containing } K$$
$$= \left\{ \sum_1^n \lambda_i x_i \;:\; x_i \in K,\ \lambda_i \geq 0,\ \sum_1^n \lambda_i = 1 \right\}.$$

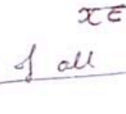
Def $x \in K$ is an extremal point of K if x is not an interior point of any interval contained in K, i.e.

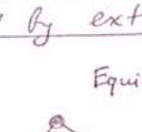
$x \notin (a,b)$ for any $a, b \in K$.

~~$x \in [a,b] = \mathrm{conv}(a,b)$, $a,b \in K$ implies $a=b=x$~~

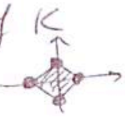
Equivalently, if $x = \lambda a + (1-\lambda) b$ for some $0<\lambda<1$ then $a=b=x$

The set of all extremal points of K is denoted by ext(K)

Examples 1)  $\mathrm{ext}(K) = \partial K$.

2)  $\mathrm{ext}(K)$ = vertices of K

3) $\mathrm{ext}(B_{\ell_\infty}) = \{(\pm 1, \pm 1, \ldots)\}$

4) $\mathrm{ext}(B_{\ell_1}) = \{(0,\ldots,0,1,0,\ldots)\}$ 

Question: Is it true that $\boxed{K = \overline{\mathrm{conv}}(\mathrm{ext}\, K)}$? in B. space X. (*)

If so, this would "simplify" K to ext K.

Another example of usefulness.

~~Proposition Assume (*) holds. Then for we~~

Proposition (Optimizing linear functions on convex sets)

Assume (*) holds, and let $f: X \to \mathbb{R}$ be a continuous convex function on X. Then

$$\sup_{x \in K} f(x) = \sup_{x \in \mathrm{ext}\, K} f(x)$$

Computationally ~~easier~~ simpler when ext K is "simpler" than K.

(Ex.)

- Unfortunately, (*) does <u>not</u> hold in general:

<u>Examples</u> 1) $\operatorname{ext}(B_{c_0}) = \emptyset$

2) $\operatorname{ext}(B_{C[0,1]}) = \{\pm 1\}$

3) $\operatorname{ext}(B_{L_1[0,1]}) = \emptyset$.

(<u>Ex.</u>)

- With compactness assumption, (*) holds:

<u>THM</u> (<u>Krein - Milman</u>)

Let K be a convex and w-compact set in a Banach space X.
Then $\operatorname{ext}(K) \neq \emptyset$. Moreover,

$$K = \overline{\operatorname{conv}(\operatorname{ext} K)}.$$

- By Alaoglu's theorem, the unit ball B_{X^*} is w^*-compact.
Hence in reflexive spaces, B_X is w-compact. Therefore, we have

<u>COR</u> If X is a reflexive Banach space, then

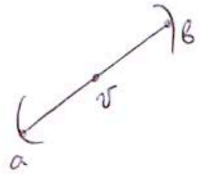
$$B_X = \overline{\operatorname{conv}(\operatorname{ext} B_X)}$$

This and the examples above show that:

<u>COR</u> c_0, $C[0,1]$, $L_1[0,1]$ are non-reflexive (also $L_\infty[0,1] = (L_1[0,1])^*$).

Proof of Krein – Milman Theorem

• Will construct an extremal point of K by contraction of extremal sets:



Def A subset $V \subseteq K$ is called an extremal subset of K

~~if every $v \in V$ satisfies: for some $a,b \in K$~~
~~& $v \in (a,b)$ implies $a,b \in V$~~
~~Denote the family of all extremal subsets of K by Ext(K)~~

if for all $v \in V$, the only way to represent $x = \lambda a + (1-\lambda) b$ for some $0<\lambda<1$ and $a,b \in K$ is to have both $a,b \in V$

~~Ext(K) is nonempty as $K \in$ Ext(K) K is an extremal subset of itself~~
~~The extremal~~

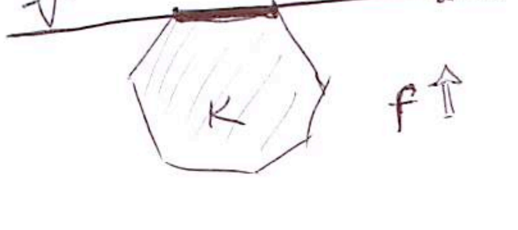
Examples of extremal sets of K:

1) the whole set K; ~~K is an extremal subset of itself~~
2) any extremal point of K, ~~is an extremal subset of K~~.
3)  For a polytope in $\mathbb{R}^n$, any face, and the whole boundary

A way to construct extremal sets.

LEMMA Let K be a convex w-compact set in X. Let $f \in X^*$, let $M = \max\limits_{x \in K} f(x)$.
Then $V = \{x \in K : f(x) = M\}$ is an extremal (convex, w-compact) subset of K.

Proof • Note that f indeed attains its maximum, because K is w-compact and f is continuous in w-topology (w-topology was defined as the weakest top. in which all $f \in X^*$ are continuous).



~~• The set is closed {f =~~

• <u>V is closed:</u> ~~hence w-compa~~ Because we can express it as an intersection of closed sets:

~~indeed,~~
$$V = K \cap \Big(\bigcap_{M' \geq M} \underbrace{\{x \in X : f(x) \leq M\}}_{\text{closed because } f \text{ is continuous.}} \Big)$$

~~Hence V is w-clo~~

• <u>V is convex</u> (also as an intersection of convex sets)

Hence V is w-closed (by Cor. p. 42)

hence <u>V is w-compact</u>.

• <u>V is extremal</u>

This is simple: ~~If $v \in V$, $v = \lambda a + (1-\lambda)b$, $a, b \in K$ ⇒ $f(v) = \lambda f(a) + (1-\lambda) f(b)$, M, $\lambda M + (1-\lambda)$~~

$\Big($ if $v \in V$, $a, b \in K$, $v = \lambda a + (1-\lambda) b$ $(0<\lambda<1)$

~~$\Rightarrow f(v) \in (f(a), f(b))$~~ $\Rightarrow$ $f(a) = f(b) = M \Rightarrow a, b \in V.$

$$\underbrace{f(v)}_{= M} = \lambda \underbrace{f(a)}_{\leq M} + (1-\lambda) \underbrace{f(b)}_{\leq M} \quad \Big)$$

QED.

• Now we need to learn how to <u>contract</u> <u>extremal sets</u> to extremal <u>points</u>.

By Zorn's lemma!

Note some simple properties of extremal sets:

(1) If A is an extremal subset of B, and B is an extremal subset of C, then A is an extremal subset of C

(2) Intersection of any number of extremal subsets of K is an extremal subset of K.

(Ex: check!)

Then we can prove a ~~weaker versi~~ the first part of K-M-T..

<u>Lemma</u> $\boxed{K \text{ has extremal points.}}$

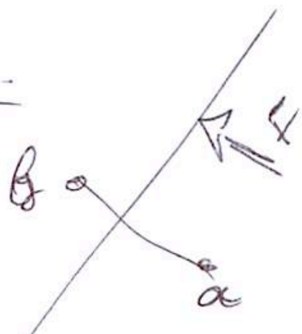
<u>Proof</u> Consider the family of all ~~closed~~ convex w-compact (⟺ closed) extremal subsets of K partially ordered by inclusion.

(This family is nonempty because K belongs there).

By property (2) on the previous page,
Zorn's lemma yields the existence of a minimal set K_0 in this family.

It suffices to prove that K_0 consists of a <u>single point</u>.

Suppose not; ~~then~~ $\alpha, \beta \in K_0$, $\alpha \neq \beta$.



Choose $f \in X^*$: $f(\beta) \neq f(\alpha)$.

By lemma p.103 applied to K_0, the set

$V = \{ x \in K_0 : f(x) = \max_{x \in K_0} f(x) \}$ is a convex w-compact extremal subset of K_0 (thus also of K).

Clearly V can not contain both α and β.

Hence V is a proper subset of K_0.

~~$\Rightarrow K_0$ is not minim~~ This contradicts minimality of K_0

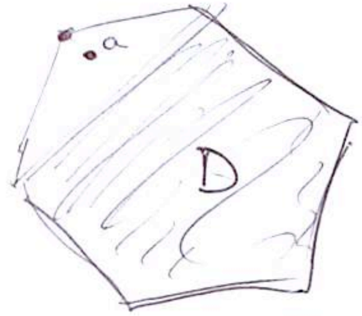
QED.

Proof of KMT

let $D = \text{conv}(\text{ext}\, K)$

$\overline{D}$ is w-compact subset of K (D closed $\Rightarrow D$ w-closed).

If $\overline{D} \neq K$, say $a \in K \setminus \overline{D}$



Separation Thm $\Longrightarrow \exists\, f \in X^*$: $f(a) > \max_{x \in \overline{D}} f(x)$

Thus $V = \{x \in K : f(x) = \max_{x \in K} f(x)\}$ is

a w-compact extremal subset of K which is disjoint from $\overline{D}$.

$\Rightarrow$ V contains an extremal point (by lemma above).
~~Thus V conta~~ This is an extremal point of K by Property (1).
This contradicts the def of D

QED

~~Ex: Prove KMT for convex w*-compact sets in X^*~~

Examples: 1) $K = \{\text{doubly stochastic matrices}\}$ - convex in $\mathbb{R}^{n^2}$. ("Birkhoff polytope")

$\text{ext}\, K = \{\text{permutations}\}$. (Ex)

Hence $\forall$ doubly stoch matrix = convex comb. of permutations (Birkhoff).

2). $K = \text{unit ball of } L(\ell_2^n, \ell_2^n)$.

$\text{ext}\, K = \{\text{unitary operators}\}$ (Ex).

$\Rightarrow \forall\, T : \ell_2^n \to \ell_2^n$, $\|T\| = 1$,

is a convex comb of unitary op's.