Krein–Milman Theorem

Recall: the convex hull of a subset $K$ of a vector space is
\[
\text{conv}(K) = \text{minimal convex set containing } K
\]
\[
= \left\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in K, \quad \lambda_i \geq 0, \quad \sum_{i=1}^{n} \lambda_i = 1 \right\}
\]

Def: $x \in K$ is an extremal point of $K$ if $x$ is not an interior point of any interval contained in $K$, i.e.,
\[
x \notin (a, b) \quad \text{for any } a, b \in K
\]
\[
x \in [a, b] = \text{conv}(a, b) \quad \text{if } a, b \in K \text{ implies } a = b = x.
\]
The set of all extremal points of $K$ is denoted by $\text{ext}(K)$.

Equivalently, if $x = \lambda a + (1-\lambda) b$ for some $0 < \lambda < 1$ then $a = b = x$.

Examples
1) $K$ is a circle
\[
\text{ext}(K) = \partial K
\]
2) $K$ is a polygon
\[
\text{ext}(K) = \text{vertices of } K
\]
3) $\text{ext}(B_1) = \{ (1,1,\ldots,1) \}$
4) $\text{ext}(B_{\infty}) = \{ (0,0,\ldots,0,1,0,\ldots) \}$

Question: Is it true that $K = \text{conv}(\text{ext}(K))$?

If so, this would "simplify" $K$ to $\text{ext}(K)$.

Another example of usefulness:

Proposition: Assume (x) holds. Then for every $x \in K$

Proposition (Optimizing linear functions on convex sets)
Assume (x) holds, and let $f : X \to \mathbb{R}$ be a continuous convex function on $X$. Then
\[
\sup_{x \in K} f(x) = \sup_{x \in \text{ext}(K)} f(x)
\]

Computationally simpler when $\text{ext}(K)$ is "simpler" than $K$. 
Unfortunately, \((*)\) does not hold in general:

**Examples**

1) \(\text{ext}(B_{c_0}) = \emptyset\)

2) \(\text{ext}(B_{c[0,1]}) = \{ \pm 1 \}\)

3) \(\text{ext}(B_{L^1[0,1]}) = \emptyset\)

With compactness assumption, \((*)\) holds:

**THM (Krein-Milman)**

Let \(K\) be a \(w\)-compact set in a Banach space \(X\). Then \(\text{ext}(K) \neq \emptyset\). Moreover,

\[ K = \text{conv}(\text{ext} K) \]

By Alaoglu's theorem, the unit ball \(B_X\) is \(w^*\)-compact. Hence in reflexive spaces, \(B_X\) is \(w\)-compact. Therefore, we have

**Cor**

If \(X\) is a reflexive Banach space, then

\[ B_X = \text{conv}(\text{ext} B_X) \]

This and the examples above show that:

**Cor**

\(c_0\), \(c[0,1]\), \(L_1[0,1]\) are non-reflexive \((\text{also } L^*_0[0,1] = (L_1[0,1])^*)\).
Proof of Krein-Milman Theorem

Will construct an extremal point of $K$ by contraction of extremal sets.

**Def.** A subset $V \subseteq K$ is called an extremal subset of $K$ if for every $v \in V$ satisfies:

- For every $v \in V$ satisfies:
  
  - For some $a, b \in K$ implies $a, b \in V$.
  
  - For every $a, b \in K$ implies $a, b \in V$.

If for all $v \in V$, the only way to represent $x = x_0 + (1 - 2)x_1$ for some $0 < x_0 < 1$ and $a, b \in K$ is to have both $a, b \in V$.

The Ext$(K)$ is non-empty so $K = \text{Ext}(K)$. $K$ is an extremal subset of itself.

Examples:

1) $K$ is the whole set $K$.

2) Any extremal point of $K$ is an extremal subset of $K$.

3) For a polyhedral in $K^n$, any facet, and the whole boundary.

A way to construct extremal sets:

Let $K$ be a convex $w$-compact set in $X$.

**Lemma.** Let $f \in X^*$, let $M = \max_{x \in K} f(x)$.

Then $\bigvee \{ x \in K : f(x) = M \}$ is an extremal subset of $K$.

Proof. Note that $f$ indeed attains its maximum, $M$ because $K$ is $w$-compact and $f$ is continuous in $w$-topology. ($w$-topology was defined as the weakest top in which all $f_{x^*}$ are continuous.)
• $V$ is closed: because we can express it as an intersection of closed sets.

\[ V = \bigcap_{M \geq M_3} \{ x \in X : f(x) \leq M \} \]

Hence $V$ is w-compact.

• $V$ is convex (also as an intersection of convex sets)

Hence $V$ is w-closed (by Cor. p. 42)

hence $V$ is w-compact.

• $V$ is extremal

This is simple:

\[ f(\beta v) = \lambda f(a) + (1-\lambda)f(b) \]

because

\[ f(a) = f(b) = M \Rightarrow a, b \in V. \]

Q.E.D.

• Now we need to learn how to contract extremal sets to extremal points

by Zorn's lemma!

Note some simple properties of extremal sets:

1. If $A$ is an extremal set of $B$, and $B$ is an extremal subset of $C$, then $A$ is an extremal subset of $C$.

2. The intersection of any number of extremal subsets of $K$ is an extremal subset of $K$.

(Ex: check!)
Lemma K has extremal points.

Proof. Consider the family of all extremal subsets of K partially ordered by inclusion. (This family is nonempty because K belongs here).

By property(2) on the previous page, Zorn's lemma yields the existence of a minimal set K₀ in this family.

It suffices to prove that K₀ consists of a single point.

Suppose not, then α, β ∈ K₀, α ≠ β.

Choose f ∈ X: f(β) ≠ f(α).

By lemma 1.5.3, applied to K₀, the set V = \{x ∈ K₀ : f(x) = f(β)\} is an extremal subset of K₀.

Clearly, V can not contain both α and β.

Hence V is a proper subset of K₀.

⇒ K₀ is not minimal. This contradicts minimality of K₀.

Q.E.D.
Proof of KMT

Let \( D = \text{conv} \text{ext}(K) \).

\( D \) is a \( \sigma \)-compact subset of \( K \).

If \( \overline{D} \neq K \), and say \( a \in K \setminus \overline{D} \).

Then \( \exists f \in X^* \colon f(a) > \max_{x \in D} f(x) \).

Thus \( V = \{ x \in K \colon f(x) = \max_{x \in K} f(x) \} \) is a \( \sigma \)-\( \sigma \)-compact extremal subset of \( K \) which is disjoint from \( \overline{D} \).

Therefore \( V \) contains an extremal point by Lemma above.

This contradicts the def of \( D \).

Q.E.D.

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**Exercise:** Prove KMT for \( \sigma \)-convex \( \sigma \)-compact sets in \( X^* \).

**Examples:**

1) \( K = \{ \text{doubly stochastic matrices} \} \) - convex in \( \mathbb{R}^{n \times n} \).
   
   "Birkhoff polytope"

   \( \text{ext} K = \{ \text{permutations} \} \).

   Hence \( V \) doubly stochastic matrix = convex comb. of permutations

   (Birkhoff).

2) \( K = \) unit ball of \( L(\ell^2, \ell^2) \).

   \( \text{ext} K = \{ \text{unitary operators} \} \).

   \( \forall T : \ell^2 \rightarrow \ell^2, \| T \| = 1 \)

   is a convex comb. of unitary ops.