Compact operators

X, Y : Banach spaces.

Def • A linear operator $T: X \to Y$ is called <u>compact</u> if it maps bounded sets to precompact sets.

• Equivalently, T is compact if it maps B_X to a precompact set.

• The set of compact operators is denoted $K(X,Y)$

Remark Compact operators are clearly bounded as precompact sets are bounded. In words,

$$K(X,Y) \subseteq L(X,Y)$$

Examples 1 (finite rank)

If $T(X)$ is a finite dimensional subspace of Y, T is called a finite rank operator.

All bounded finite rank operators are compact. (because $T(B_X)$ is a bounded subset of a f.d. space $T(X)$ $\Rightarrow$ precompact)

~~Ex Suppose T can be approximated by finite rank operators $T_n : X \to Y$.~~
$\|T - T_n\| \to 0$ as $n \to \infty$.
Show that T is compact.

Example 2 (integral operators)

$T: C[0,1] \to C[0,1]$ defined as

$$(Tf)(t) = \int_0^1 k(t,s) f(s)\,ds$$

where $k(t,s) \in C([0,1] \times [0,1])$.

Example 2 (Integral operators)

$T: C[0,1] \to C[0,1]$ defined as

$$(Tf)(t) = \int_0^1 k(t,s)\, f(s)\, ds$$

where $k(t,s) \in C([0,1]^2)$ is a kernel.

We claim that T is compact

It suffices to check that $T(B_{C[0,1]})$ is a precompact set of $C[0,1]$

~~By Arzéla thm, it suffices to check that that set is (uniformly) bounded and equicontinuous.~~

Use Arzéla thm:

1) (Uniform) boundedness follows from the boundedness of T. (see Example p. 65).

2) Equicontinuity. Let $\varepsilon > 0$; $\exists\, \delta > 0$ such that

$$|t_1 - t_2| \le \delta \text{ implies } |k(t_1,s) - k(t_2,s)| \le \varepsilon.$$

Hence for every $f \in B_{C[0,1]}$,

$$|(Tf)(t_1) - (Tf)(t_2)| \le \int_0^1 |k(t_1,s) - k(t_2,s)|\, \underbrace{|f(s)|}_{\le 1}\, ds \le \varepsilon.$$

QED

• Compact operators are usually not invertible!

~~Proposition~~ ~~The identity operator on an ∞-dim. Banach space is not compact (because the unit ball is not compact – Riesz's Thm p. 90).~~

~~Cor Isomorphism operators are not compact (same reason).~~

Example 3 Let X be an ∞-dim. Banach space. Then every isomorphism $T: X \to Y$ is not compact. In particular, the identity operator on X is not compact

(follows from the non-compactness of B_X – Thm p. 90).

THM (Properties of the set of compact operators $K(X,Y)$).

(i) $K(X,Y)$ is a closed linear subspace of $\mathcal{L}(X,Y)$;

(ii) $K(X,Y)$ is an operator ideal. ~~in $\mathcal{L}(X,Y)$~~

This means that if $T \in K(X,Y)$ then the compositions ST and TS are compact for every bounded linear operator S.

Proof (i) Linearity follows from the observation that the sum of two precompact sets is precompact; a scalar dilate of a precompact set is precompact (Ex: Check!)

Closedness Let $T_n \in K(X,Y)$, ~~$T \in \mathcal{L}(X,Y)$~~,

$$T_n \to T \text{ in } \mathcal{L}(X,Y).$$

Let $\varepsilon > 0$, choose $n \in \mathbb{N}$ s.t. $\|T_n - T\| \le \varepsilon$.

For every $x \in B_X$, $\|T_n x - Tx\| \le \varepsilon$.

This shows that $T_n(B_X)$ is an ε-net of $T(B_X)$.

Since $T_n(B_X)$ is a precompact set, it follows from this that $T(B_X)$ is also precompact (check!)

QED

Cor Suppose $T: X \to Y$ can be approximated by finite rank operators $T_n: X \to Y$, i.e. $\|T_n - T\| \to 0$ as $n \to \infty$.

Then T is compact.

Example (Diagonal op.) $T: \ell_2 \to \ell_2$: $Tx = (\lambda_i x_i)_1^\infty$

~~where $\lambda_i \to 0$.~~

Ex: prove that T is compact iff $\lambda_i \to 0$

~~(b) prove that $\lambda_i \to 0$ is also necessary for T being compact.~~

Recall: $T \in L(X,Y) \Rightarrow T^* \in L(Y^*, X^*)$; $\|T^*\| = \|T\|$

THM (Schauder) Let X, Y be Banach spaces, and $T \in K(X,Y)$

Then $T^* \in K(Y^*, X^*)$.

• Before the proof, we ~~mentions~~ make some observation that motivates the ~~next~~ argument (but will not be used).

Prop $C(K)$ is a universal space. Every B. space X can be isometrically embedded into $C(K)$ for some top. compact space K.

Proof $K := (B_{X^*}, w^* \text{ topology})$; define the embedding $X \hookrightarrow C(K)$ by associating $x \in X$ the function $x(f) := f(x)$, $f \in K$. (i.e. $x \mapsto (f(x))_{f \in K}$).

~~This is a linear isometric emb~~

This map is obviously linear. It is an isometry:

$$\|x\|_{C(K)} = \max_{f \in K} |f(x)| = \|x\|_X.$$

QED

Proof of Schauder's thm.

~~It suffices to prove that the set $G := T^*$~~

We know that $K = \overline{T(B_X)}$ is compact in Y. We want to show: $G = T^*(B_{Y^*})$ is precompact in X^*.

We will embed G into $C(K)$ and use Arzela's theorem.

So, we define the embedding $U: G \to C(K)$ by

$$U(T^*f) := f|_K \quad \text{for } f \in B_{Y^*}. \qquad \text{(some selection)}$$

1) U is an isometric embedding:

$$\|T^*f\|_{X^*} = \sup_{x \in B_X} (T^*f)(x) = \sup_{x \in B_X} f(Tx) = \sup_{y \in K} f(y) = \|f|_K\|_{C(K)}.$$

2) $U(G)$ is uniformly bounded in $C(K)$:

$$\|f|_K\|_{C(K)} = \|T^*f\|_{X^*} \le \|T^*\| \underbrace{\|f\|_{Y^*}}_{\le 1} \le \|T\|.$$

~~recall $\|T$~~

3) $U(G)$ is equicontinuous: for $y_1, y_2 \in K$,

~~$|f|_K(y_1 - y_2)| = f(y$~~ $$|f|_K(y_1) - f|_K(y_2)| = |f(y_1 - y_2)| \le \underbrace{\|f\|_{X^*}}_{\le 1} \|y_1 - y_2\|.$$

Hence f is 1-Lipschitz.

QED