LECTURE 27

Fredholm theory

- It studies operators of the form $I - T$ "identity plus compact" where I is the identity on a Banach space X, $T: X \to X$ is a compact operator.

- Motivations: 2) solving linear equations $\lambda x - Tx = b$
  1) spectral theory: $\lambda I - T$.

- $\approx I$ - finite rank op.

$$\boxed{\textbf{THM 1}\quad \text{Let } K \in \mathcal{K}(X,X). \text{ Then the image of } I - T \text{ is closed.}}$$

Will follow from a criterion of closedness:

Prop (Closed image). Let $A \in L(X,Y)$ where X, Y are Banach spaces.

(a) Suppose A is bounded below, i.e. $\exists c > 0$:
$$\|Ax\| \geq c\|x\| \quad \forall x \in X.$$
(Stronger than injectivity)
Then A has closed image. Hence A is isomorphism between X and $\operatorname{Im} A \leq Y$.

(b) Conversely, if A is injective and has closed image then A is bounded below.

Proof (a) Denoting $C = \|A\|$, we have
$$c\|x\| \leq \|Ax\| \leq C\|x\|, \quad x \in X.$$
This means that A is an isomorphism btw (normed spaces) X and $\operatorname{Im} A \leq Y$.
Since X is complete $\Rightarrow \operatorname{Im} A$ is complete $\Rightarrow$ closed.

(b) When considered as $A: X \to \operatorname{Im} A^{\leq Y}$ (B. spaces), the operator A is injective and surjective.
By I.M.T, A is an isomorphism.

QED

Proof of THM p. 111.

- Let $A = 1 - T$; ~~consider the injectivi~~

  consider the injectivization $\tilde{A}$ of A:

$$\begin{tikzcd} X \arrow[r, "A"] \arrow[d, "q"'] & X \\ X/\mathrm{Ker}\,A \arrow[ur, "\tilde{A}"'] & \end{tikzcd}$$

  Since $\mathrm{Im}\,A = \mathrm{Im}\,\tilde{A}$, by Prop (closed image)

  it suffices to show that $\underline{\tilde{A} \text{ is bounded below}}$.

- Suppose not: $\exists\ [x_n]$, $\|[x_n]\| = 1$ such that $\tilde{A}[x_n] \to 0$.

  $\Rightarrow \exists$ representatives $x_n \in X$, $\|x_n\| \le 2$ such that

$$\mathrm{dist}\,(x_n, \mathrm{Ker}\,A) = 1 \quad \text{but} \quad \underbrace{Ax_n}_{= x_n - Tx_n} \to 0.$$

- By compactness of T, we may assume that $Tx_n \to z \in X$.

  Hence $\underline{x_n \to z}$.

  Since $Ax_n \to 0 \Rightarrow Ax_n \to \boxed{Az = 0} \Rightarrow \underline{z \in \mathrm{Ker}\,A}$

  It follows that $\mathrm{dist}\,(x_n, \mathrm{Ker}\,A) \to 0$, Contradiction. QED

(Fredholm)

THM Let $T \in K(X,X)$.

Then the operator $I-T$ is injective if and only if it is surjective.

- Remark: 1) should be familiar from linear algebra in finite dimensional spaces.
  2) Does not hold for non-compact op's in general:
  right shift in l_2 is ~~not~~ injective but not surjective;
  left shift in l_2 is surjective but not injective.

- Remark Thm clearly holds for $\lambda I - T$, $\lambda \neq 0$.

An interpretation of Thm p. 113 yields the following

- Fredholm Alternative for linear equations. Let $T \in K(X,X)$.

  Then, either the equation $\lambda x - Tx = 0$ has a nontrivial solution (non-inj)

  or the equation $\lambda x - Tx = b$ has a solution for all $b \in X$.

- Fredholm used this alternative for integral operators $Tx(t) = \int_0^1 x(s) k(t,s) ds$

which are, as we have seen, compact:

To study the integral equation

$$\lambda x(t) - \int_0^1 x(s) k(t,s) ds = b(t), \qquad (*)$$

One studies the homogeneous integral equation

$$\lambda x(t) - \int_0^1 x(s) k(t,s) ds = 0. \qquad (**)$$

~~Eq (*) has~~

If (**) does not have nontrivial solutions then (*) has unique sol. $\forall b$.

Proof of sufficiency in Thm.

- Suppose that $A := I - T$ is injective but <u>not</u> surjective.

$$Y_n := \mathrm{Im}(A^n), \qquad n = 0, 1, 2, \dots$$

- Claim. $Y_0 \supset Y_1 \supset Y_2 \supset \dots$ are proper inclusions

Indeed, $Y_0 = X \supset \mathrm{Im}\, A = Y_1$ is a proper inclusion. The injective operator A preserves proper inclusions, hence $Y_1 = A(Y_0) \supset A(Y_1) = Y_2$ etc.

- Claim. Y_n are closed subspaces.

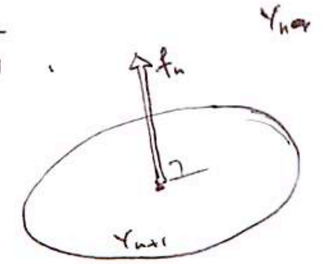
Indeed, $\underbrace{A^n = (I-T)^n = I - T_1}_{\text{by the binomial expansion,}}$ for some compact operator T_1.

The claim then follows from Thm p.111.

- By Hahn-Banach theorem, we can find

$$f_n \in Y_n^* \text{ such that } \|f_n\| = 1, \quad f_n \in Y_{n+1}^\perp.$$

(Ex: check!)



We can extend f_n so that $f_n \in X^*$.
(again by Hahn-Banach)

- We will show that $(T^* f_n)$ contains no convergent subsequences.

This will contradict the compactness of T^* and, by Schauder's thm, the compactness of T.

To this end, for $n > m$ we compute

$$\|T^* f_n - T^* f_m\| = \|T^*(f_n - f_m)\|$$

$$= \|\underbrace{(I^* - T^*)(f_n - f_m) + f_m - f_n}\|$$

$$\geq \sup_{x \in B_{Y_n}} |\langle \cdot, x \rangle|$$

$$= \sup_{x \in B_{Y_n}} \Big| \underbrace{\langle \underbrace{f_n - f_m}_{\in Y_{n+1}^\perp}, \underbrace{(I-T)x}_{= Ax \in Y_{n+1}} \rangle}_{=0} + \langle \underbrace{f_m}_{\in Y_n^\perp} - f_n, \underbrace{x}_{\in Y_n} \rangle \Big|$$

$$= \sup_{x \in B_{Y_n}} |\langle f_n, x \rangle| = 1. \qquad \text{QED.}$$