Spectrum of bounded linear operators.

1. Finite dimensional space case

Let \( \mathbf{T} \): linear op. on \( \mathbb{C}^n \).
We identify \( \mathbf{T} \) with its \( n \times n \) matrix.

Def. \( \lambda \in \mathbb{C} \) is an eigenvalue of \( \mathbf{A} \) if \( \ker(\mathbf{T} - \lambda \mathbf{I}) \neq 0 \), i.e. \( \exists x \in \mathbb{C}^n \) such that
\[
\mathbf{T} x = \lambda x.
\]
\( x \) is called an eigenvector of \( \mathbf{A} \) associated with the eigenvalue \( \lambda \).

Properties
- \( \mathbf{A} \) always has eigenvalues
- Eigenvectors corresponding to different eigenvalues are linearly independent (but may not form a basis - the dimension of the space of eigenvectors corresponding to an eigenvalue may be strictly less than the multiplicity of that root). e.g. for \( \mathbf{T} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \), a Jordan block.
- Eigenvectors form an orthonormal basis.
- There exists an orthonormal basis of eigenvectors iff \( \mathbf{A} \) is normal, i.e. \( \mathbf{A}^* \mathbf{A} = \mathbf{A} \mathbf{A}^* \).
2. Infinite dimensional spaces.

- Example 1: ODE \( u' = \lambda u \)

\( \lambda \) is an eigenvalue of the differential op. \( T = \frac{d}{dt} \)
acting on the space of differentiable functions.
Solving this ODE gives the eigenvectors
\[ u(t) = Ce^{\lambda t} \]
Hence every \( \lambda \in \mathbb{C} \) is an eigenvalue of \( T \). (uncountable!)

- Example 2: Multiplication operator in \( L_2(0,1) \) defined as
\[ (Tx)(t) = tx(t) \]
Any eigenvalues \( \lambda \) of \( T \) must satisfy the equation
\[ tx(t) = \lambda x(t) \]
for some \( x \in L_2(0,1) \) and all \( t \in (0,1) \)

i.e.
\[ (t-\lambda)x(t) = 0 \]
\[ \Rightarrow \ x(t) = 0 \]
Hence \( \ker (T - \lambda I) = 0 \), and \( T \) has no eigenvalues.
Definition and classification of spectrum.

Let $T \in L(X,X)$ where $X$ is a Banach space.

$\lambda \in \mathbb{C}$ is called a regular point if $T - \lambda I$ is invertible, i.e., $(T - \lambda I)^{-1} \in L(X,X)$.

Otherwise we call $\lambda$ a spectrum point.

The set of all spectrum points is denoted $\sigma(T)$ and called the spectrum of $T$.

Definition (Classification).

- The point spectrum $\sigma_p(T)$ is the set of all eigenvalues of $T$.
  
  i.e., the set of $\lambda \in \mathbb{C}$ satisfying
  
  $\ker(T - \lambda I) \neq \emptyset$,
  
  $\dim \ker(T - \lambda I)$ is called the multiplicity of $\lambda$.

- The continuous spectrum $\sigma_c(T)$ is the set of all $\lambda \in \mathbb{C}$ such that
  
  $\ker(T - \lambda I) = \emptyset$ and $\text{Im}(T - \lambda I)$ is dense in $X$
  
  (note that $\text{Im}(T - \lambda I) \neq X$ otherwise $T - \lambda I$ would be invertible by the closed graph theorem).

- The residual spectrum $\sigma_r(T)$ is the set of all $\lambda \in \mathbb{C}$ satisfying
  
  $\ker(T - \lambda I) = \emptyset$ and $\text{Im}(T - \lambda I)$ is not dense in $X$.

Therefore, the spectrum $\sigma(T)$ can be expressed as a disjoint union

$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$. 
Example 1: diagonal operators in $l_2$

$$T((x_i)_{i=1}^\infty) = (\lambda_i x_i)_{i=1}^\infty,$$

where $\lambda_i \rightarrow 0$ is some sequence in $\mathbb{C}\setminus\{0\}$.

1) As $(T-\lambda I)x = ((\lambda_i - \lambda)x_i)_{i=1}^\infty$,

we have $(T-\lambda I)^* y = \left(\frac{1}{\lambda_i - \lambda} y_i \right)_{i=1}^\infty$.

It follows that $(T-\lambda I)^*$ is a bounded linear operator if

$\lambda$ is not in the closure of $\{\lambda_i\}_{i=1}^\infty$, which is $\{\lambda_i\} \cup \{0\}$.

2) $\lambda_i$ are clearly the eigenvalues of $T$ as $Te_i = \lambda_i e_i$

3) $0$ is not an eigenvalue as $T$ is injective ($\lambda_i \neq 0$)

But $\text{Im}(T)$ is dense in $l_2$ (as $\lambda_i \neq 0$),

Hence $0$ is in the continuous spectrum of $T$.

$$\sigma_p(T) = \{\lambda_i\}_{i=1}^\infty; \quad \sigma_c(T) = \{0\}; \quad \sigma_r(T) = \varnothing.$$
Example 2: multiplication operator in $L_2(0,1)$

$$(Tx)(t) = tx(t).$$

1) As $(T-\lambda I)x(t) = (t-\lambda)x(t)$,
we have

$$(T-\lambda I)^{-1} y(t) = \frac{1}{t-\lambda} y(t) \quad (1)$$

a) If $\lambda \notin [0,1]$ then the function $\frac{1}{t-\lambda}$ is bounded
and $(T-\lambda I)^{-1}$ is a bounded operator.

Therefore, such $\lambda$ are regular points.

b) If $\lambda \in (0,1)$ then $\frac{1}{t-\lambda} \notin L_2(0,1)$ (non-integrable singularity at $\lambda$)
and $(T-\lambda I)^{-1}$ is not invertible (at $y(t) = 1$).

Hence such $\lambda$ are spectrum points.

Therefore, $\sigma(T) = [0,1]$.

2) As we already noticed on p. 117, $T$ has no eigenvalues.

3) It follows from $(\ast)$ that $\Im (T-\lambda I)$ is dense in $L_2[0,1]$.

(Indeed, given $z(t) \in L_2[0,1]$ and $\varepsilon > 0$,

define $y(t) = z(t) \Pi \{ |t-\lambda| > \varepsilon \}$, then $(T-\lambda I)^{-1} y(t) = \frac{1}{t-\lambda} y(t) \in L_2[0,1]$).

Conclusion: $\sigma_p(T) = \emptyset$; $\sigma_c(T) = (0,1)$; $\sigma_r(T) = \emptyset$.

Remark: If a delta function
If Dirac's delta function was a genuine function, $\delta_0 \in L_2[0,1]$,
then clearly $T\delta_0 = \lambda \delta_0$ so $\delta_0$ would be eigenfunctions of $T$.
Example 3: Shift operators in $l^2$

\[ S_R \left( x_1, x_2, x_3, \ldots \right) = (0, x_1, x_2, x_3, \ldots) \]
\[ S_L \left( x_1, x_2, x_3, \ldots \right) = (x_2, x_3, \ldots) \]

1) $e \in \sigma_r(S_R)$. Proof

Indeed, $S_R$ is not surjective (first coordinate is not in the image).

Indeed, $S_R$ is clearly injective but $\text{Im } S_R$ is not dense in $l^2$ ($e \in 1 \text{ Im } S_R$).

In fact,

\[ \sigma_p(S_R) = \emptyset, \quad \sigma_c(S_R) = \{ \lambda : |\lambda| = 1^\frac{1}{2} \}, \quad \sigma_r(S_R) = \{ \lambda : |\lambda| \leq 1^\frac{1}{2} \}. \]

2) $\sigma_p(S_L) = \{ \lambda : |\lambda| < 1^\frac{1}{2} \}, \quad \sigma_c(S_L) = \{ \lambda : |\lambda| = 1 \}, \quad \sigma_r(S_L) = \emptyset$.

Proposition: $\lambda \in \sigma(T)$ if and only if $\lambda \in \overline{\sigma(T)}$ where $\overline{\sigma(T)}$ stands for complex conjugation (not closure!)

(Ex).

Deuality of point spectrum and residual spectrum (Exo)