Spectrum of bounded linear operators.

(1) Finite dimensional ~~space~~ case

. Let T: linear op. on $\mathbb{C}^n$.

We identify T with its $n \times n$ matrix.

Def. $\lambda \in \mathbb{C}$ is an <u>eigenvalue</u> of A if $\ker(T - \lambda I) \neq 0$,

i.e. $\exists\, x \in \mathbb{C}^n$ such that

$$Tx = \lambda x.$$

x is called an <u>eigenvector</u> of A associated with the eigenvalue λ.

~~The set of all eigenvectors is called the spectrum of A.~~

Properties.

. A always has eigenvalues ~~A is not empty~~ (eigenvalues are the roots of the characteristic polynomial $\det(T - \lambda I) = 0$, which has roots by the Fund. Th. of Alg.)

. Eigenvectors corresponding to different eigenvalues are <u>linearly independent</u> (but may not form a basis — the dimension of the space of eigenvectors corresponding to an eigenvalue (the root of the char. poly) may be strictly less than the multiplicity of that root), — e.g. for $T = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, a Jordan block).

~~. Eigenvectors form an orthonorm~~

. There exists an <u>orthonormal basis of</u> eigenvectors iff A is <u>normal</u>, i.e. $A^*A = AA^*$.

② Infinite dimensional spaces.

- Example 1: ODE $\boxed{u' = \lambda u}$

λ is an eigenvalue of the differential op. $T = \frac{d}{dt}$

acting on the space of differentiable functions.

Solving this ODE gives the eigenvectors

$$u(t) = c e^{\lambda t}.$$

Hence every $\lambda \in \mathbb{C}$ is an eigenvalue of T. (uncountable!)

- Example 2 Multiplication operator in $L_2[0,1)$ defined as

$$(Tx)(t) = t\,x(t).$$

An eigenvalue λ of T must satisfy the equation

$$t\,x(t) = \lambda x(t) \quad \text{for some } x \in L_2[0,1) \text{ and all } t \in (0,1)$$

i.e. $(t-\lambda)\,x(t) = 0$

$\Rightarrow x(t) = 0.$

Hence $\ker(T - \lambda I) = 0$, and T has no eigenvalues.

General

Definition and classification of spectrum.

Def Let $T \in L(X,X)$ where X is a Banach space.

$\lambda \in \mathbb{C}$ is called a <u>regular point</u> if $T-\lambda I$ is invertible, i.e. $(T-\lambda I)^{-1} \in L(X,X)$.

Otherwise we call λ a <u>spectrum point</u>.

The set of all spectrum points is denoted $\sigma(T)$ and called the <u>spectrum</u> of T.

Def (Classification).

- The <u>point spectrum</u> $\sigma_p(T)$ is the set of all eigenvalues of T, i.e. the set of $\lambda \in \mathbb{C}$ satisfying
$$\ker(T-\lambda I) \neq 0.$$
$\dim \ker(A-\lambda I)$ is called the <u>multiplicity</u> of λ.
- The <u>continuous spectrum</u> $\sigma_c(T)$ is the set of all $\lambda \in \mathbb{C}$ such that
$$\ker(T-\lambda I) = 0 \quad \text{and} \quad \mathrm{Im}(T-\lambda I) \text{ is } \underline{\text{dense}} \text{ in } X$$
(note that $\mathrm{Im}(T-\lambda I) \neq X$ otherwise $T-\lambda I$ would be invertible by PMT).
- The <u>residual spectrum</u> $\sigma_r(T)$ is the set of all $\lambda \in \mathbb{C}$ satisfying
$$\ker(T-\lambda I) = 0 \quad \text{and} \quad \mathrm{Im}(T-\lambda I) \text{ is } \underline{\text{not dense}} \text{ in } X.$$

Therefore, the spectrum $\sigma(T)$ can be expressed as a disjoint union
$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T).$$

• Example 1: diagonal operators in l_2

$$T((x_i)_{i=1}^{\infty}) = (\lambda_i x_i)_{i=1}^{\infty}$$

where $\lambda_i \to 0$ is some sequence in $\mathbb{C}\setminus\{0\}$.

1) As $(T-\lambda I)x = ((\lambda_i - \lambda)x_i)_{i=1}^{\infty}$,

we have $(T-\lambda I)^{-1} y = \left(\frac{1}{\lambda_i - \lambda} y_i\right)_{i=1}^{\infty}$.

It follows that $(T-\lambda I)^{-1}$ is a bounded linear operator if

λ is not in the closure of $\{\lambda_i\}_{i=1}^{\infty}$, which is $\{\lambda_i\} \cup \{0\}$.

2). λ_i are clearly eigenvalues of T as $Te_i = \lambda_i e_i$

3) 0 is not an eigenvalue as T is injective $(\lambda_i \neq 0)$

But $\mathrm{Im}(T)$ is dense in l_2 (as $\lambda_i \neq 0$).

Hence 0 is in the continuous spectrum of T.

Conclusion: $\sigma_p(T) = \{\lambda_i\}_{i=1}^{\infty}$; $\sigma_c(T) = \{0\}$; $\sigma_r(T) = \emptyset$.

- <u>Example 2</u> : multiplication operator in $L_2[0,1]$.

$$(Tx)(t) = t\,x(t).$$

1) As $(T-\lambda I)x(t) = (t-\lambda)\,x(t)$,

we have $\boxed{(T-\lambda I)^{-1} y(t) = \frac{1}{t-\lambda}\, y(t)}$ $\qquad (*)$

a) If $\lambda \notin [0,1]$ then the function $\frac{1}{t-\lambda}$ is bounded,

and ~~thus~~ $(T-\lambda I)^{-1}$ is a bounded operator.

Therefore, such λ are <u>regular points</u>

b) If $\lambda \in [0,1]$ then $\frac{1}{t-\lambda} \notin L_2[0,1]$ (non-integrable singularity at λ)

$T-\lambda I$ is not invertible (at $y(t)=1$).

Hence such λ are <u>spectrum point</u>.

Therefore, $\underline{\sigma(T) = [0,1]}$.

2). As we already noticed on p. 117, T has <u>no eigenvalues</u>.

3) It follows from $(*)$ that $\operatorname{Im}(T-\lambda I)$ is dense in $L_2[0,1]$.

(Indeed, given $z(t) \in L_2[0,1]$ and $\varepsilon > 0$,
define $y(t) = z(t)\, \mathbb{1}_{\{|t-\lambda| > \varepsilon\}}$; then $(T-\lambda I)^{-1} y(t) = \frac{1}{t-\lambda}\, y(t) \in L_2[0,1]$).

- <u>Conclusion</u> : $\sigma_p(T) = \emptyset$; $\quad \sigma_c(T) = [0,1]$; $\quad \sigma_r(T) = \emptyset$

- <u>Remark</u> : ~~If a delta funct~~ If Dirac's delta function was a genuine function, $\delta_\lambda \in L_2[0,1]$,
then clearly $T\delta_\lambda = \lambda \delta_\lambda \Rightarrow \delta_\lambda$ would be <u>eigenfunctions of T</u>
(as in the discrete case).

• Example 3 : shift operators in l_2 :

$$S_R(x_1, x_2, x_3 \dots) = (0, x_1, x_2, x_3 \dots)$$

$$S_L(x_1, x_2, x_3 \dots) = (x_2, x_3, \dots)$$

1) $0 \in \sigma_r(S_R)$.

~~Indeed, S_R is not surjective (first coordinate = 0 in the image)~~

Indeed, S_R is clearly injective
but $\mathrm{Im}\, S_R$ is not dense in l_2 ($e_1 \perp \mathrm{Im}\, S_R$).

In fact,

$\sigma_p(S_R) = \emptyset$, $\sigma_c(S_R) = \{\lambda : |\lambda| = 1\}$, $\sigma_r(S_R) = \{\lambda : |\lambda| < 1\}$.

2) $\sigma_p(S_L) = \{\lambda : |\lambda| < 1\}$, $\sigma_c(S_L) = \{\lambda : |\lambda| = 1\}$, $\sigma_r(S_L) = \emptyset$.

Proposition: $\lambda \in \sigma(T)$ iff $\overline{\lambda} \in \overline{\sigma(T)}$ where $\overline{\ }$ here stands for complex conjugation (not closure!)

(Ex).

Duality of point spectrum and residual spectrum (Ex)