LECTURE 29

~~Resolvent Properties of spectrum~~

Resolvent, and Properties of spectrum

Let $T \in L(X,X)$, $T \neq 0$.

Def $\rho(T) := \mathbb{C} \setminus \sigma(T)$ is called the resolvent set.

$R(\lambda) = R_T(\lambda) = (T - \lambda I)^{-1}$ is called the <u>resolvent</u> of T.

It is a function $\rho(T) \to L(X,X)$.

Lemma (Von Neumann) Let $\|S\| < 1$. Then $I - S$ is invertible,

$$(I-S)^{-1} = \sum_{k=0}^{\infty} S^k, \qquad \|(I-S)^{-1}\| \leq \frac{1}{1-\|S\|}.$$

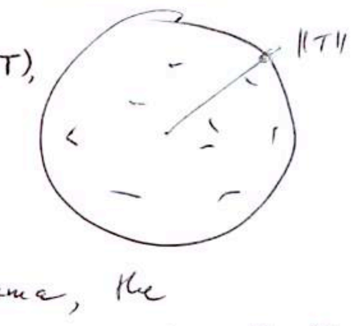
Proof

- The series $\sum_{k=0}^{\infty} S^k$ converges because $\|S\| < 1$
- $(I-S)\sum_{k=0}^{\infty} S^k = \left(\sum_{k=0}^{\infty} S^k\right)(I-S) = I$ is easily checked.
- $\|(I-S)^{-1}\| \leq \sum_{k=0}^{\infty} \|S^k\| \leq \sum_{k=0}^{\infty} \|S\|^k \leq \frac{1}{1-\|S\|}$.

QED

Prop $\sigma(T)$ is a bounded set; $\sigma(T) \subset \{\lambda : |\lambda| \leq \|T\|\}$

Proof Let $|\lambda| > \|T\|$; it suffices to show that $\lambda \in \rho(T)$, i.e. that $T - \lambda I$ is invertible.



$$T - \lambda I = \lambda\left(\underbrace{\tfrac{1}{\lambda}T - I}\right).$$

If $\|\frac{1}{\lambda} T\| < 1$ then by Von Neumann's lemma, the operator is invertible. QED.

→ **Remark**: Moreover from Von Neumann's: $|\lambda| > \|T\|$ implies

$$\boxed{R(\lambda) = -\sum_{k=0}^{\infty} \lambda^{-k-1} T^k;}$$

$\boxed{\|R(\lambda)\| \leq \frac{1}{\lambda - \|T\|}}$

More on this in "Spectral radius"

~~COR Remark If $|\lambda| > \|T\|$ then $R(\lambda) = -\sum_{k=0}^{\infty} \lambda^{-k-1} T^k$~~ ~~Proof (directly from~~

Prop (Resolvent's main identity). For $\lambda, \mu \in \rho(T)$,

$$\boxed{R(\lambda) - R(\mu) = (\lambda - \mu)\, R(\lambda)\, R(\mu).}$$

Proof ~~For scalar~~ In the scalar case, the identity holds:

$$\frac{1}{x-\lambda} - \frac{1}{x-\mu} = \frac{x-\mu-x+\lambda}{(x-\lambda)(x-\mu)} = \frac{\lambda-\mu}{(x-\lambda)(x-\mu)}$$

It is easy to extend this argument to ~~easy to verify it~~ in the operator case, ~~too~~ (check!) QED

- ~~This idea~~ Solve this ~~idea~~ equation for $R(\mu)$:

$$R(\mu) = [(\lambda-\mu)R(\lambda) - I]^{-1} R(\lambda) \qquad (*)$$

~~If this operator~~

~~If $(\lambda-\mu)R(\lambda) - I$ is invertible then~~

- It follows from this argument that, if $\lambda \in \rho(T)$ and the operator $(\lambda-\mu)R(\lambda) - I$ is invertible then $R(\mu)$ is a bounded linear op expressed as $(*)$, hence $\mu \in \rho(T)$.

  Using Von Neumann's lemma, we conclude that this happens in the neighborhood $\{\mu : |\lambda - \mu| < \frac{1}{\|R(\lambda)\|}\}$

  Hence we have proved:

Cor $\rho(T)$ is an open set. Equivalently, $\sigma(T)$ is an ~~o~~ closed set.

- Hence: $\sigma(T)$ is a compact set.

~~**Cor** $R(\lambda)$ is an analytic ~~continuous~~ function $\mathbb{C} \to L(X,X)$ on its domain $\rho(T)$.~~

~~**Proof** by $(*)$ and Von Neumann, $R(\mu) = \sum_{k=1}^{\infty} (\mu-\lambda)^{k-1} R(\lambda)^{-k}$ $(**)$~~

~~Hence $R(\mu)$ can be represented by a convergent power series centered at any pt. $\lambda \in \rho(T)$. QED~~

If we apply (in (x) p. 123) von Neumann's expansion to the operator $(\lambda-\mu)R(\lambda)-I$, we deduce:

Cor The resolvent $R(\lambda)$ is an analytic operator-valued function on its domain $\rho(T)$. Specifically, ~~for any~~ $R(\mu)$ can be expressed as a convergent power series ~~centered at any~~ in a small neighborhood of any point $\lambda\in\rho(T)$:

$$\boxed{R(\mu)=\sum_{k=1}^{\infty}(\mu-\lambda)^{k-1}R(\lambda)^{-k}} \qquad (**)$$

□.

Remark It follows that $\forall f\in L(X,X)^*$, $f(R(\mu))$ is an analytic function $\mathbb{C}\to\mathbb{C}$.

THM The spectrum $\sigma(T)$ of any operator $T\in L(X,X)$ is nonempty.

Proof • Assume that $\sigma(T)=\emptyset$; hence $\rho(T)=\mathbb{C}$, and $R(\lambda)$ is then an entire function (i.e. analytic on $\mathbb{C}$).

• $R(\lambda)$ is also a bounded function on $\mathbb{C}$, with $R(\lambda)\to 0$ as $\lambda\to\infty$.

(Indeed, $\|R(\lambda)\|\le\frac{1}{\lambda-\|T\|}$ for $|\lambda|>\|T\|$ (see the Remark on p. 122).

~~Boundedness o~~ This proves that $R(\lambda)\to 0$ as $\lambda\to\infty$, and that $R(\lambda)$ is bounded on $|\lambda|>\|T\|$. Because $R(\lambda)$ is continuous (by Cor. above), $R(\lambda)$ is also bounded on $|\lambda|\le\|T\|$.)

• By Liouville's theorem, $R(\lambda)\equiv\text{const}$ (hence $R(\lambda)\equiv 0$).

(Indeed, we can apply Liouville's theorem to the analytic function

$$\mu\mapsto f(R(\mu)),\qquad \text{where } f\in L(X,X)^* \text{ is } \forall \text{ fixed}$$

Then $f(R(\mu))\equiv 0\quad\forall f\in L(X,X)^*\Rightarrow R(\mu)\equiv 0$.)

• This contradicts the invertibility of $R(\mu)$. QED

Remark: We have shown: $\boxed{\sigma(T)\text{ is a }\cancel{\text{comp}}\text{ nonempty compact set.}}$