LECTURE 3

5) $C(K)$, where K is a compact topological space (think of $K = [a,b]$).

$C(K)$ consists of all continuous functions $f : K \to \mathbb{R}$ (or $\mathbb{C}$)

with the norm

$$\|f\|_\infty := \max_{x \in K} |f(x)| \qquad \text{(max attained)}.$$

Exercise: check norm axioms.

6) $L_1(\Omega, \Sigma, \mu)$, where (Ω, Σ, μ) is a measure space.

(think of $(\Omega, \Sigma, \mu) = (a,b)$)

as defined before, & L_1 is the space of all integrable functions on Ω

with the norm

$$\|f\|_1 := \int_\Omega |f(x)|\, d\mu \quad \left(=: \int |f|\, d\mu.\right)$$

Exercise: check the norm axioms.

Remark: For $(\Omega, \Sigma, \mu) = (\mathbb{N}, \text{counting measure})$,

$L_1(\Omega, \Sigma, \mu) = \ell_1$, with the same norm $\|\cdot\|_1$.

8) $1 \le p < \infty$; $L_p(\Omega, \Sigma, \mu)$ consists of all "p-integrable" functions on Ω, i.e. for which

$$\text{exists} \quad \int_\Omega |f(x)|^p\, dx < \infty.$$

The norm is defined as

$$\|x\|_p := \left(\int_\Omega |f(x)|^p\, d\mu\right)^{1/p}.$$

Ran $L_p = \ell_p$

Norm axioms (i), (ii) straightforward; (iii) follows from

Minkowski inequality (for functions) $\forall\, 1 \le p < \infty$, $\forall\, f, g \in L_p(\Omega, \Sigma, \mu)$.

$$\left(\int |f(x) + g(x)|^p\, d\mu\right)^{1/p} \le \left(\int |f(x)|^p\, d\mu\right)^{1/p} + \left(\int |g(x)|^p\, d\mu\right)^{1/p}$$

Convexity of norms and balls

Def (Balls) ~~in norm~~ Let X be a normed space.

(Closed) Ball centered at $x_0 \in X$ with radius r:

$$B_X(x_0, r) := \{ x \in X : \|x - x_0\| \leq r \}$$

(Closed) unit ball:

$$B_X := B_X(0,1) = \{ x \in X : \|x\| \leq 1 \}$$

Unit sphere:

$$S_X = \{ x \in X : \|x\| = 1 \}.$$

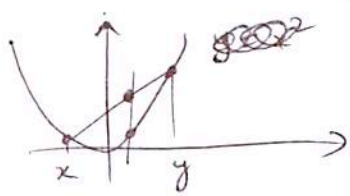
Exercise B_X is a closed set in X $\quad (x_n \in B_X,\ x_n \to x \Rightarrow x \in B_X)$.

Def (Convexity) ~~Recall~~: 1) A function $f: E \to \mathbb{R}$ defined on a linear space is convex if

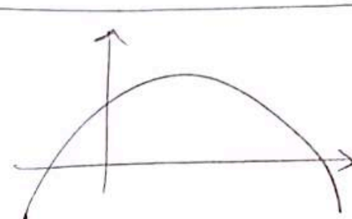
$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y). \qquad \begin{pmatrix} \forall x, y \in E; \\ \forall \lambda \in [0,1] \end{pmatrix}$$

2) A set $K \subset E$ is convex if: $x, y \in K \Rightarrow \lambda x + (1-\lambda) y \in K \quad \forall \lambda \in [0,1]$

Examples 1) $E = \mathbb{R}$

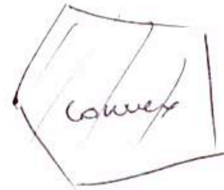


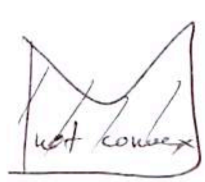
convex



not convex (concave).

2)





__Proposition__ (norms and Balls are convex). Let X be a normed space.

1) The function $x \mapsto \|x\|$ is convex on X
2) The unit ball B_X is a closed, symmetric convex set ($x \in B_X \Rightarrow -x \in B_X$). ~~(so are all balls $B(x_0,r)$)~~

1) $\|\lambda x + (1-\lambda) y\| \le \lambda \|x\| + (1-\lambda)\|y\|$ by the norm axioms.

2) If $x, y \in B_X$ then from (1):

$\|\lambda x + (1-\lambda) y\| \le \lambda + (1-\lambda) = 1 \Rightarrow \lambda x + (1-\lambda) y \in B_X$. QED

Symmetry follows from norm axiom (ii).

Reverse direction:

__Proposition__ (convex functions are norms, their level sets ~~convex sets~~ are Balls).

Let $x \mapsto \|x\|$ be a function on a ~~normed sp~~ linear space E which satisfies axioms (i) and (ii) of a norm.

1) If $x \mapsto \|x\|$ is a convex function then it ~~satisf it is a norm on~~ ~~satis~~ it is a norm on E (i.e. satisfies Δ ineq).
2). If the level set $\{x \in E : \|x\| \le 1\}$ is convex then $\|\cdot\|$ is a norm on E

(1) ~~Use $\|\lambda x$~~ Follows from $\|\lambda x + (1-\lambda) y\| \le \lambda\|x\| + (1-\lambda)\|y\|$ with $\lambda = 1/2$: $\|x+y\| \le \|x\| + \|y\|$.

(2) Convexity of B means: if ~~$u, v \in B_X$, $\lambda \in (0,1) \Rightarrow \lambda u +$~~ If $\|u\| \le 1$, $\|v\| \le 1$, $\lambda \in [0,1] \Rightarrow \|\lambda u + (1-\lambda) v\| \le 1$. $(*)$

Let $x, y \in E$; we want to show that $\|x+y\| \le \|x\| + \|y\|$.

$\Updownarrow$

~~First make~~ ~~norm-one vectors:~~ $\left\| \frac{x}{\|x\|+\|y\|} + \frac{y}{\|x\|+\|y\|} \right\| \le 1$.

Use $(*)$ for $u = \frac{x}{\|x\|}$, $v = \frac{y}{\|y\|}$, $\lambda = \frac{\|x\|}{\|x\|+\|y\|} \Rightarrow$ QED.

Example: L_p, = space of p-integrable functions.

let $1 \le p < \infty$, (Ω, Σ, μ) measure space

$L_p = L_p^{(\Omega,\Sigma,\mu)}$ is defined as the space of all functions $f : \Omega \to \mathbb{R}$ s.t.

$$\|f\|_p := \left(\int_\Omega |f(x)|^p \, d\mu \right)^{1/p} < \infty .$$

Probably, have come across L_2 before.

Proposition $\|\cdot\|_p$ is a norm on $L_p(\Omega, \Sigma, \mu)$.

(i), (ii) are straightforward; (iii) (Δ ineq.) is not.

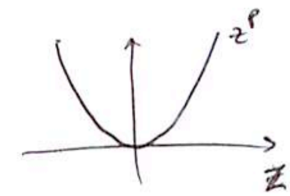
We will instead check that the level set

$$B_p := \{ f \in L_p : \|f\|_p \le 1 \}$$

is convex, and finish by Prop. on p 12.

let $f, g \in B_p$, let $\lambda \in [0,1]$.

The function $z \to z^p$ is convex on $\mathbb{R}$ $(p \ge 1)$



$\Rightarrow \forall \ t \in \Omega$ we have an inequality

$$|\lambda f(t) + (1-\lambda) g(t)|^p \le \lambda |f(t)|^p + (1-\lambda) |g(t)|^p .$$

Integrating yields

$$\int |\lambda f + (1-\lambda) g|^p \, d\mu \le \lambda \underbrace{\int |f|^p d\mu}_{\le 1} + (1-\lambda) \underbrace{\int |g|^p d\mu}_{\le 1} \le 1 .$$

$\Rightarrow \|\lambda f + (1-\lambda) g\|_p \le 1 \Rightarrow \lambda f + (1-\lambda) g \in B_p.$ QED.

Corollary (Minkowski inequality)

$\forall \ 1 \le p < \infty$, $\forall \ f, g \in L_p(\Omega, \Sigma, \mu)$, one has

$$\left(\int |f(t) + g(t)|^p \, d\mu \right)^{1/p} \le \left(\int |f(t)|^p d\mu \right)^{1/p} + \left(\int |g(t)|^p d\mu \right)^{1/p}$$

i.e. $\|f + g\|_p \le \|f\|_p + \|g\|_p$