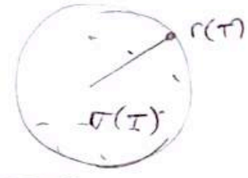
Spectral radius



- We know that $\sigma(T) \subseteq \{\lambda : |\lambda| \le \|T\|\}$
  but a much stronger ~~inclusion is known~~ bound holds.

Def The spectral radius $r(T) := \max\{|\lambda| : \lambda \in \sigma(T)\}$

THM $r(T) = \lim\limits_{n} \|T^n\|^{1/n} = \inf\limits_{n} \|T^n\|^{1/n}$.

~~2) For $|\lambda| > r(T)$, the resolvent can be expressed as power series~~
~~$R(\lambda) = -\sum_{k=0}^{\infty} \lambda^{-k-1} T^k$~~

Proof ~~We clai~~

1) Claim: If $\lambda \in \sigma(T)$ then $\lambda^n \in \sigma(T^n)$.

(This is a partial case of the "Spectral Mapping Thm": $p(\sigma(T)) = \sigma(p(T))$ $\forall$ polynomial p)

Indeed, we can factor $T^n - \lambda^n I^n = S(T - \lambda I)$

where $S = (T^{n-1} + T^{n-2}(\lambda I) + \dots + (\lambda I)^{n-1})$.

(similar to $a^n - b^n = (a^{n-1} + a^{n-2}b + \dots + b^{n-1})(a-b)$).

~~Hence~~ Therefore, if $T^n - \lambda^n I^n$ is invertible then $T - \lambda I$ is invertible.

This proves the claim.

- ~~Since $\sigma(T^n) \subseteq \{$~~ By Claim and Prop. p. 122,
  $\lambda \in \sigma(T)$ implies $|\lambda^n| \le \|T^n\| \Rightarrow |\lambda| \le \|T^n\|^{1/n}$.

Hence $r(T) \le \inf \|T^n\|^{1/n}$. $(*)$

~~let $|\lambda| > r(T)$. Then by what we $(*)$ we have~~

Exercise: Let $S, T \in \mathcal{L}(X, X)$. Prove that ST is invertible iff both S and T are invertible.

2). We know two things: (for every $f \in L(X,X)^*$):

a) the function $f(R(\lambda))$ is an ~~holomorphic~~ analytic function on the annulus $|\lambda| > r(T)$; (Cor p.124)
~~(see Remark)~~

b) the function $f(R(\lambda))$ is represented by a Laurent series

$$(*) \qquad f(R(\lambda)) = -\sum_{k=0}^{\infty} \lambda^{-k-1} f(T^k) \qquad \text{in the smaller annulus } |\lambda| > \|T\|$$

(Remark p.122)

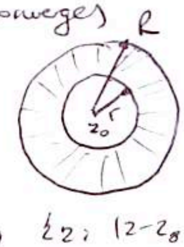
By the theory of convergence of Laurent series of analytic functions, the series (*) converges in the ~~annulus~~ larger annulus $|\lambda| > r(T)$.

<u>Quote</u>: Consider a Laurent series

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n .$$

- There exists unique $r, R \in \mathbb{R} \cup \{\infty\}$ such that Laurent series converges in the annulus

$$A = \{ z : r < |z - z_0| < R \}$$



and diverges outside the closure of A.

- Moreover, $\exists$ at least one point on the inner boundary $\{z : |z - z_0| = r\}$ of A and on the outer boundary $\{z : |z - z_0| = R\}$ of A such that $f(z)$ can <u>not</u> be analytically continued to those points.

Hence the terms of the series are bounded:

$$\sup_n |\lambda^{-n-1} f(T^n)| < \infty \quad \text{for every } |\lambda| > r(T) \text{ ~~and every~~}$$

(Cor. p.83)

Since $f \in L(X,X)^*$ is arbitrary, the Principle of Uniform Boundedness implies that

$$\sup_n \|\lambda^{-n-1} T^n\| =: K < \infty$$

$\Rightarrow$ Taking n-th root $\Rightarrow |\lambda|^{-1-\frac{1}{n}} \|T^n\|^{1/n} \le K^{1/n} \quad \forall n$

$$\Rightarrow \limsup_n \|T^n\|^{1/n} \le 1/|\lambda| .$$

Since this holds for all $|\lambda| > r(T)$, we conclude that

$$\boxed{\limsup_n \|T^n\|^{1/n} \le r(T)} .$$

This and part (1) implies:

$$r(T) \le \inf \|T^n\|^{1/n} \le \liminf \|T^n\|^{1/n} \le \limsup \|T^n\|^{1/n} \le r(T). \qquad \text{QED}$$

Spectrum of compact operators

Consider $T \in K(X,X)$.

$\boxed{\textbf{Prop} \quad 0 \in \sigma(T)}$

(This is a reformulation of the fact (p.108) that compact operators are not invertible).

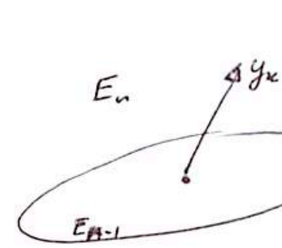
Prop (point spectrum). ~~The poin~~

- For every $\varepsilon > 0$ there exists a finite number of linearly independent eigenvectors corresponding to eigenvalues λ_i with $|\lambda_i| > \varepsilon$.
- Consequently, the point spectrum $\sigma_p(T)$ ~~lies in a sequence~~ is at most countable, and it lies in a sequence that converges to 0.

Every eigenvalue λ_i has finite multiplicity: $\dim \ker(T - \lambda_i I) < \infty$.

Proof • Suppose ~~there~~ for some $\varepsilon > 0$ there exists an infinite sequence of linearly independent vectors $(x_i)_1^\infty$ such that

$$T x_i = \lambda_i x_i, \quad |\lambda_i| > \varepsilon.$$



• Let $E_n = \operatorname{span}(x_i)_1^n$; then $E_1 < E_2 < \dots$ (proper inclusions).

Choose $y_n \in E_n$, $\|y_n\| = 1$ such that $\operatorname{dist}(y_n, E_{n-1}) \geq \frac{1}{2}$ $\left(\begin{array}{c}\text{Ex: why?} \\ \text{Consider } E_n / E_{n-1}\end{array}\right)$.

We will show that $(Ty_n)_1^\infty$ contains no Cauchy subsequences ($\Rightarrow T$ not compact ↯)

To this end, we express y_n in the form

$$y_n = \sum_1^n a_i x_i = a_n x_n + \underbrace{\sum_1^{n-1} a_i x_i}_{\in E_{n-1}}$$

Then

$$T y_n = \lambda_n a_n x_n + \underbrace{\sum_1^{n-1} \lambda_i a_i x_i}_{\in E_{n-1}}$$

For $n > m$, we have

$$\|Ty_n - Ty_m\| = \|\lambda_n a_n x_n + \underbrace{u_{n,m}}_{\in E_{n-1}}\| = \|\lambda_n y_n + \underbrace{v_{n,m}}_{\in E_{n-1}}\| \geq \lambda_n \operatorname{dist}(y_n, E_{n-1}) \geq \varepsilon/2.$$

QED

$\boxed{\textbf{THM} \quad \sigma(T) = \sigma_p(T) \cup \{0\}}$ Proof: Let $\lambda \neq 0$. By Fredholm Alternative (p.113), either $T - \lambda I$ is not injective (hence $\lambda \in \sigma_p(T)$) or $T - \lambda I$ is injective and surjective (hence $\lambda \notin \sigma(T)$).

QED