Spectral of self-adjoint operators.

- Let $H$ be a Hilbert space, $T \in L(H, H)$

Recall from Lecture 17 that $T \in L(H, H)$: the adjoint of $T^* \in L(H, H)$ is defined as $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for $x, y \in H$.

Def: $T \in L(H, H)$ is called a self-adjoint operator if $T^* = T$, i.e., $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in H$.

1) Operators on $C^n$ given by Hermitian matrices $a_{ij} = \overline{a_{ji}}$.

Examples: Integral operators $(Tf)(t) = \int K(s,t) f(s) \, ds$ with symmetric kernel $K(s,t) = \overline{K(t,s)}$.

3) Orthogonal projections $P$ on $H$.

$\sigma(P) = \sigma_r(P) = \{0, 1\}$ (Check)

Consider the self-adjoint $T \in L(H, H)$, $\sigma(T)$

Its quadratic form is $Jy(x) = \langle Tx, x \rangle$ on $H$.

Since $y(x) = \langle Tx, x \rangle = \langle x, T^*x \rangle = \langle T^*x, x \rangle = y(x)$,

the quadratic form takes only real values.

Quadratic form:

- It is possible to show that $J$ is a quadratic form.

Polarization identity:

$\langle Tx, y \rangle = \langle T$
Bilinear and quadratic forms

Let $T \in L(\mathbb{H}, \mathbb{H})$. Then
$$g(x, y) = \langle x, Ty \rangle$$
is a bilinear form.

Generally,

**Def:** A map $g : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{C}$ is called a bilinear form if

- (i) $g(a_1 x_1 + a_2 x_2, y) = a_1 g(x_1, y) + a_2 g(x_2, y)$
- (ii) $g(x, a_1 y_1 + a_2 y_2) = \bar{a}_1 g(x, y_1) + \bar{a}_2 g(x, y_2)$.

**Ex:** In $\mathbb{C}^n$, every bilinear form has the form
$$g(x, y) = \sum_{i,j=1}^n a_{ij} x_i y_j.$$

Every $T \in L(\mathbb{H}, \mathbb{H})$ defines a bilinear form. Converse is also true:

**Prop (Bilinear forms):** Let $g(x, y)$ be a bilinear form in $\mathbb{H}$, which is continuous separately in $x$ and in $y$.

Then $\exists$ unique $T \in L(\mathbb{H}, \mathbb{H})$ such that
$$g(x, y) = \langle x, Ty \rangle$$
for all $x, y \in \mathbb{H}$.

**Proof:** For every $x \in \mathbb{H}$, for a fixed $y \in \mathbb{H}$, $g(x, y)$ is a continuous linear functional on $\mathbb{H}$. By Riesz Rep. Thm, $g(x, y) = \langle x, Ty \rangle$.

It remains to check linearity and continuity of $T$:

1) **Linearity:** Simple

2) **Continuity:** For every $x \in \mathbb{H}$, $\langle x, Ty \rangle = g(x, y)$ is continuous in $y$.

Hence $y_n \rightarrow 0$ implies $\langle x, Ty_n \rangle \rightarrow 0$ by P.U.B., $(Ty_n)$ is a bounded sequence.

Therefore $T$ maps sequences converging to $0$ to bounded seq's. Hence $T$ is bounded. (Check!)
Def. A map \( h : H \to C \) is called a quadratic form if
\[
\forall x \in H \quad h(x) = g(x, x) \quad \text{for some bilinear form}.
\]

So, a bilinear form \( g \) defines a quadratic form \( h(x) \)

Conversely, a quadratic form can be defined by \( g(x, y) \).

The bilinear form \( g \) can be reconstructed from the quadratic form by
the polarization identity

\[
\forall x, y \in H \quad g(x, y) = \frac{1}{4} \left[ h(x + y) - h(x - y) + i h(x + iy) - i h(x - iy) \right]
\]

(Check!)

(Similar to the parallelogram law) (for \( \mathbb{C} \))

(Reconstructs the norm \( \|x\| \) from the inner product \( \langle x, y \rangle \) in \( H \))

- Let \( A \in L(H, H) \) be a self-adjoint operator.

Its quadratic form is then

\[
h(x) = \langle Ax, x \rangle
\]

(Reconstructed takes real values only)

Remark: One can reconstruct \( A \) from its quadratic form.

(Indeed, the quadratic form \( h \) defines the bilinear form \( \langle y, x \rangle \) by \( x \));

which in turn defines \( A \) by Prop 2.129.

2) \( h(x) \) only takes real values.

\[
\forall x \in H \quad h(x) = \langle Ax, x \rangle = \langle x, Ax \rangle = \langle x, x \rangle = \|x\|^2
\]

(For real \( H \))
One can conveniently use the quadratic form $\langle Tx, x \rangle$ to compute $\|T\|$: 

**Theorem**

Let $T \in L(H, H)$ be a self-adjoint operator. Then

$$\|T\| = \sup_{x \in S_H} |\langle Tx, x \rangle|.$$ 

**Proof**

$\subseteq$: For every $x \in S_H$,

$$|\langle Tx, x \rangle| \leq \|Tx\| \cdot \|x\| \leq \|T\| \cdot \|x\|,$$

(Cauchy-Schwarz)

$\supseteq$: Use polarization identity. Write

$$\|T\| = \sup_{x, y \in S_H} |\langle Tx, y \rangle| = \sup_{x, y \in S_H} \Re \langle Tx, y \rangle,$$

and use polarization identity

$$\langle Tx, y \rangle = \frac{1}{4} \left[ \langle Tx+y, x-y \rangle - \langle T(x-y), x-y \rangle \right];$$

denote $M = \sup_{x \in S_H} |\langle Tx, x \rangle|$

$$\leq \frac{M}{4} \left[ \|x+y\|^2 + \|x-y\|^2 \right].$$

Use parallelogram identity:

$$= \frac{2M}{4} \left( \|x\|^2 + \|y\|^2 \right) \leq M.$$ 

Q.E.D.