LECTURE 31

## ~~Spectrum of~~ Self-adjoint operators.

- Let $H$ be a Hilbert space, $T \in \mathcal{L}(H,H)$

Recall from lecture 17 that ~~$T \in \mathcal{L}(H,H)$~~ the adjoint op. $T^* \in \mathcal{L}(H,H)$ is defined as

$$\langle Tx, y\rangle = \langle x, T^* y\rangle \quad \text{for } x, y \in H.$$

**Def** $T \in \mathcal{L}(H,H)$ is called a self-adjoint operator if $T^* = T$, i.e.

$$\langle Tx, y\rangle = \langle x, Ty\rangle \quad \text{for all } x, y \in H.$$

Examples:
1) Operators on $\mathbb{C}^n$ given by Hermitian matrices: $a_{ij} = \overline{a_{ji}}$.
2) integral operators $(Tf)(t) = \int_0 K(s,t) f(s)\, ds$ with symmetric kernels $K(s,t) = \overline{K(t,s)}$.
3) Orthogonal projections $P$ on $H$; $\sigma(P) = \sigma_p(P) = \{0, 1\}$. (Check!)

Lemma $\forall A \in \mathcal{L}(H,H)$ can be uniquely represented as $A = T + iS$; $T, S$ self adj.

Proof If $A = T + iS$ then $A^* = T - iS$ $\Rightarrow T = \frac{A + A^*}{2}$; $S = \frac{A - A^*}{2i}$ QED.

Quadratic form.

- Consider a self-adjoint $T \in \mathcal{L}(H,H)$.

**Def** Its quadratic form is $g(x) = \langle Tx, x\rangle$ on $H$.

Since $g(x) = \langle Tx, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle} = \overline{g(x)}$,

the quadratic form takes only real values.

- Quadratic form

- ~~It is possible to show that a quadratic form~~

Polarization identity

$\langle Tx, y\rangle = \langle T$

# Bilinear and quadratic forms

- Let $T \in L(H,H)$. Then

$$g(x,y) = \langle x, Ty \rangle \quad \text{is a bilinear form.}$$

Generally:

**Def** A map $g : H \times H \to \mathbb{C}$ is called a bilinear form if

$$\text{(i)} \quad g(a_1 x_1 + a_2 x_2, y) = a_1 g(x_1, y) + a_2 g(x_2, y)$$

$$\text{(ii)} \quad g(x, a_1 y_1 + a_2 y_2) = \bar{a}_1 g(x_1, y_1) + \bar{a}_2 g(x_2, y_2)$$

**Ex.** In $\mathbb{C}^n$, every bilinear form has the form

$$g(x,y) = \sum_{i,j=1}^{n} a_{ij} x_i \bar{y}_j .$$

~~2) Every~~

Every $T \in L(H,H)$ defines a bilinear form. Converse is also true:

**Prop** (Bilinear forms) Let $g(x,y)$ be a bilinear form in $H$, which is continuous separately in $x$ and in $y$.

Then $\exists_{\text{unique}} T \in L(H,H)$ such that

$$g(x,y) = \langle x, Ty \rangle \quad \text{for all } x, y \in H.$$

**Proof** ~~For every $x$ 1) linearity:~~ For a fixed $y \in H$, $g(x,y)$ is a continuous linear functional on $H$. By Riesz Repr. Thm, $\exists$unique $T(y) \in H$ s.t.

$$g(x,y) = \langle x, Ty \rangle .$$

It remains to check linearity and continuity of $T$.

1) linearity - simple

2) Continuity: For every $x \in H$, $\langle x, Ty \rangle = g(x,y)$ is continuous in $y$ $\Rightarrow Ty_n \xrightarrow{\text{is weakly}} 0 \Rightarrow Ty_n$ weak-bounded

Hence $y_n \to 0$ implies $\langle x, Ty_n \rangle \to 0$. By P.U.B., $(Ty_n)$ is a bounded sequence.

Therefore $T$ maps sequences converging to 0 to bounded seq's. Hence $T$ is bounded. (Check!)

**Def** A map ~~g(x)~~ $h : H \to \mathbb{C}$ is called a <u>quadratic form</u> if $h(x) = g(x,x)$ for some bilinear form.

- So, a bilinear form $g(x,y)$ defines a quadratic form $h(x)$. Conversely, ~~a quadratic forms can be reco~~ the bilinear form $g(x,y)$ can be reconstructed from the quadr form $h(x)$ by the <u>polarization identity</u>

$$g(x,y) = \frac{1}{4}\left[ h(x+y) - h(x-y) + i h(x+iy) - i h(x-iy) \right] \quad \text{(Check!)} \quad (*)$$

(~~Similar to the parallelogram law!~~). ~~(See~~ (Similar to the identity that reconstructs the norm $\|\cdot\|_2$ from the inner product $\langle \cdot , \cdot \rangle$ on $H$)

- Let $T \in L(H,H)$ be a self-adjoint operator. Its quadratic form is then

$$\boxed{h(x) = \langle Tx, x \rangle}$$

~~h(x) takes real values only~~

**Remark** 1) <u>One can reconstruct $T$ from its quadratic form.</u>
(indeed, the quadratic form $\langle Tx, x\rangle$ defines the bilinear form $\langle Tx, y\rangle$ by $(*)$; which in turn defines $T$ by Prop p.129)

2) <u>$h(x)$ only takes real values.</u>
$$\left( h(x) = \langle Tx, x \rangle = \langle x, Tx \rangle = \overline{\langle Tx, x \rangle} = \overline{h(x)} . \right)$$

One can conveniently use the quadratic form $\langle Tx, x\rangle$ to compute $\|T\|$:

**THM** Let $T \in L(H,H)$ be a self adjoint op. Then

$$\|T\| = \sup_{x \in S_H} |\langle Tx, x\rangle| .$$

**Proof** ("$\leq$"): For every $x \in S_H$,

$$|\langle Tx, x\rangle| \leq \|Tx\| \cdot \underbrace{\|x\|}_{=1} \leq \|T\| .$$

(Cauchy-Schwarz)

($\geq$): ~~Use polarization identity:~~ ~~Write~~ Use polarization identity:

$$\|T\| = \sup_{x,y \in S_H} |\langle Tx, y\rangle| = \sup_{x,y\in S_H} \operatorname{Re}\langle Tx, y\rangle ;$$

~~and use polarization identity~~

$$\langle Tx, y\rangle = \frac{1}{4}\left[\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle\right]$$ ; denote $M = \sup_{x \in S_H} |\langle Tx, x\rangle|$

$$\leq \frac{M}{4}\left[\|x+y\|^2 + \|x-y\|^2\right].$$ Use parallelogram identity:

$$= \frac{2M}{4}\left[\|x\|^2 + \|y\|^2\right] \leq M.$$

QED