Spectrum of self-adjoint operators

Prop. A self-adjoint operator $T \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ does not have a residual spectrum.
\[ \sigma_r(T) = \emptyset \]

(see Problem 5 HW 11/13).

Prop (Criterion of invertibility) Let $T \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ be a self-adjoint operator.

1) $T$ is invertible if and only if $T$ is bounded below, i.e. \( \exists c > 0 : \|Tx\| \geq c\|x\| \) for all $x \in \mathcal{H}$.

2) Consequently, $\lambda \in \sigma(T)$ if and only if the operator $(T - \lambda I)$ is not bounded below.

Proof. Recall the criterion of closed image (p.111):
If $T$ is bounded below then $\text{Im} T$ is closed. But $0 \notin \sigma_r(T) = \emptyset$, hence $\text{Im} T$ is dense in $\mathcal{H}$. Hence $\text{Im} T = \mathcal{H}$, hence $T$ is surjective and injective (becausebdd below).
\[ \Rightarrow T \text{ is invertible} \quad \square \]

Thm (Spectrum of self-adjoint operators)

Let $T \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ be a self-adjoint operator. Then:

1) The spectrum $\sigma(T)$ is real and in fact
\[ \mathbb{R} \subseteq \sigma(T) \subseteq [m, M] \subseteq [-\|T\|, \|T\|] \]
where
\[ m = \inf_{x \in \mathcal{H}} \langle Tx, x \rangle \quad \text{and} \quad M = \sup_{x \in \mathcal{H}} \langle Tx, x \rangle \]

2) The endpoints $m, M \in \sigma(T)$.

Remark. Recall from p.120 that $\sigma(T) \cap \mathbb{R}$ for self-adjoint $T$. 

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Proof: 1. Let $\lambda \in \mathbb{C} \setminus \{m, M\}$;

For $x \in x_M$, let's bound the norm

$$
\| (T - \lambda) x \| \geq | \langle (T - \lambda) x, x \rangle | \quad \text{by C-S}
$$

$$
= | \langle Tx, x \rangle - \lambda \langle x, x \rangle | \\
= | \langle Tx, x \rangle - \lambda | \geq d.
$$

Hence $T - \lambda I$ is bounded below $\Rightarrow \lambda \notin \sigma(T)$ by Prop. p. 132.

2. Let's prove $M \in \sigma(T)$.

\[\text{WLOG, } 0 \leq m \leq M.\]

By translation:

(Indeed, otherwise consider $T - M I$ instead of $T$; i.e. $T - M I \leq [0, M - m]$; hence prove that $M - m \in \sigma(T - M I) \Rightarrow M \in \sigma(T)$)

Hence $M = \sup \| Tx, x \| = \| T \|$. 

Choose $x_n \in x_M$ s.t.

$$
\langle Tx_n, x_n \rangle \to M
$$

Then

$$
\| (T - M I) x_n \|^2 = \langle (T - M I) x_n, (T - M I) x_n \rangle
$$

$$
= \| T x_n \|^2 - 2M \langle T x_n, x_n \rangle + M^2 \langle x_n, x_n \rangle
$$

$$
= \frac{\| T x_n \|^2 - 2M M + M^2 M}{M^2} \to 0
$$

$\exists \text{limit } (T - M I) x_n M = 0 \Rightarrow T - M I \text{ is not bounded below } \Rightarrow M \in \sigma(T)$.

By Prop. (Entero p. 130) \& Prop. (T)

$M \in \sigma(T)$. Consider $-T + M I$. Then $\sigma(-T + M I) \subseteq [0, M - m]$; hence $M - m \notin \sigma(-T + M I)$.

$\therefore M \in \sigma(T)$. Q.E.D.
Example: \( T \) is the linear transformation of \( T \).

\[ T(\mathbf{v}) = \mathbf{w} \]

for a surface \( E \) is called an invariant surface of \( T \).

Let \( \mathbf{v} = \mathbf{x}_1 \) or \( \mathbf{v} = \mathbf{x}_2 \).

Then \( \mathbf{x}_1 = \mathbf{x}_2 \).

Theorem (Compact Self-Adjoint Operators)

Special Theorem: The compact self-adjoint operators

Proof: (Let \( T \) be a self-adjoint operator. Then the eigenvalues converge to zero.)
Prop. Let $T \in L(K, K)$ be self-adjoint.

If $E \subseteq K$ is an invariant subspace of $T$,
then $E^\perp$ is also an invariant subspace of $T$.

Proof. Let $x \in E^\perp$, we need to show that $Tx \in E^\perp$.

Let $y \in E^\perp$; then
\[ \langle Tx, y \rangle = \langle x, Ty \rangle = 0. \]
QED

Lemma. Let $T \in L(K, K)$ be a compact self-adjoint operator, $K \neq \{0\}$. Then $T$ has eigenvalue $0$.
Then $T$ has nonzero eigenvectors.

Thm. (Spectral theorem for compact self-adjoint operators)

Let $T$ be a compact self-adjoint linear operator
on a separable Hilbert space $H$.

Then there exists an orthonormal basis $(\{e_n\})_{n \in \mathbb{N}}$ in $H$ consisting of eigenvalues of $T$.

Proof. 1) Let's first prove that $T$ has a non-zero eigenvector.

Recall that since $T$ is compact,
\[ \sigma(T) = \sigma_p(T) \cup \{0\} \quad (\text{Thm. p.127}) \]

If $\sigma_p(T) \neq \{0\}$ then $T$ has nonzero eigenvectors.

If $\sigma_p(T) = \{0\}$ then $\sigma(T) = \{0\}$ by Cr p.134, $\|T\| = \tau(T) = 0 \Rightarrow T = 0$

$\Rightarrow$ every vector in $K$ is an eigenvector of $T$.

2) Consider the family of all orthonormal sequences $\{e_n\}$ in $K$ consisting of eigenvectors of $T$.

By Zorn's lemma, it has a maximal sequence $\{e_n\}$. Let $E = \text{span}(\{e_n\})$. Suppose $E \neq K$.

Clearly $E$ is an invariant subspace of $T$; hence $E^\perp$ is also an invariant subspace (by Prop).

Using Part 1 for the restriction of $T$ onto $E^\perp$, we conclude that $E^\perp$ contains an eigen vector $v$ of $T$, but this contradicts the maximality of $\{e_n\}$.

QED
Remark (Diagonal Form)

- \( \mathbb{R}^n \). Since \( T h_n = \lambda_n h_n \) for the orthonormal basis \((h_n)\) of \( H \), we have:

\[
T \text{ is a diagonal operator in the orthonormal basis of its eigenvectors.}
\]

- Recall that

\[
x = \sum_{n=1}^{\infty} \langle x, h_n \rangle h_n , \quad \text{for } x \in H
\]

Then

\[
T x = \sum_{n=1}^{\infty} \langle x, h_n \rangle \lambda_n h_n
\]

\[
T x = \sum_{n=1}^{\infty} \lambda_n \langle x, h_n \rangle h_n.
\]

Remark (Non-separable spaces \( H \))

- We never used (in a crucial way) that \( H \) is separable. If \( H \) is a general Hilbert space, the argument above yields a similar version of (4) for a possibly uncountable orthonormal basis \((h_\alpha)\) (i.e. \( \langle h_\alpha, h_\beta \rangle = \delta_{\alpha\beta} \); \( \text{span} \{h_\alpha\} = H \)).

- Since there are only countably many \( h_\alpha \) corresponding to nonzero eigenvalues \( \lambda_\alpha \) (by compactness of \( T \)), one can deduce that (4) still holds for the orthonormal system \((h_n)\) consisting of the \( \lambda_n \) eigenvectors corresponding to nonzero eigenvalues \( \lambda_n \).