Spectrum of selfadjoint operators

Prop A selfadjoint operator $T \in L(H,H)$ does not have a residual spectrum:

$$\sigma_r(T) = \emptyset$$

(see Problem 5 HW 11/13).

Prop (Criterion of invertibility) Let $T \in L(H,H)$ be a self-adjoint operator.

1). T is invertible if and only if T is bounded below, i.e. $\exists\, C>0$ s.t.:

$$\|Tx\| \geq c\|x\| \text{ for all } x \in H.$$

2) Consequently, $\lambda \in \sigma(T)$ if and only if the operator $(T-\lambda I)$ is not bounded below.

Proof • Recall the criterion of closed image (p.111):

if T is bounded below then $\operatorname{Im} T$ is closed (converse also true if T is injective).

• But $0 \notin \sigma_r(T) = \emptyset$, hence $\operatorname{Im} T$ is dense in H.

Hence $\operatorname{Im} T = H$. Hence T is surjective and injective (because bdd below)

$\Rightarrow$ T is invertible

QED

THM (Spectrum of self-adjoint operators)

Let $T \in L(H,H)$ be a self-adjoint operator. Then:

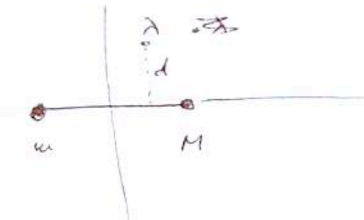
1) The spectrum $\sigma(T)$ is real and in fact

$$\sigma(T) \subseteq [m, M] \subseteq [-\|T\|, \|T\|]$$

where $$m = \inf_{x \in S_H} \langle Tx, x\rangle \;; \quad M = \sup_{x \in S_H} \langle Tx, x\rangle$$

2) The endpoints $m, M \in \sigma(T)$

Remark. Recall from p.130 that $\langle Tx, x\rangle \in \mathbb{R}$ for self-adj T



Proof 1). Let $\lambda \in \mathbb{C}\setminus[m,M]$;

$d := \mathrm{dist}(\lambda,[m,M]) > 0$.

For $x\in S_H$, let's bound the norm

$$\|(T-\lambda I)x\| \geq |\langle (T-\lambda I)x, x\rangle| \qquad \text{by C-S}$$

$$= |\langle Tx,x\rangle - \lambda\langle x,x\rangle|$$

$$= |\underbrace{\langle Tx,x\rangle}_{\in [m,M]} - \lambda| \geq d.$$

Hence $T-\lambda I$ is bounded below $\Rightarrow \lambda\notin\sigma(T)$ by Prop. p. 132.

2). Let's prove $M\in\sigma(T)$.

• W.L.O.G., $0\le m\le M$.

(By translation: indeed, otherwise consider $T-MI$ instead of T; ~~its spectrum $\subseteq$~~ $\sigma(T-mI)\subseteq[0,M-m]$; prove that $M-m\in\sigma(T-mI) \Rightarrow M\in\sigma(T)$)

~~Hence $M = \sup\langle Tx,x\rangle = \|T\|$~~

~~Hence $M = \|T\|$.~~ Hence $M=\sup\{\langle Tx,x\rangle\} = \|T\|$.

Choose $x_n\in S_H$ s.t.

$$\langle Tx_n,x_n\rangle \to M \qquad (n\to\infty).$$

Then

$$\|(T-MI)x_n\|^2 = \langle (T-MI)x_n, (T-MI)x_n\rangle$$

$$= \underbrace{\|Tx_n\|^2}_{\leq \|T\|^2 = M^2} - 2M\underbrace{\langle Tx_n,x_n\rangle}_{\to M} + M^2\underbrace{\|x_n\|^2}_{=1} \to 0$$

~~$\leq 2M^2 - 2M\langle Tx_n,x_n\rangle$~~

$\Rightarrow \liminf \|(T-MI)x_n\|^2 = 0$. $\Rightarrow$ $T-MI$ is not bounded below $\Rightarrow M\in\sigma(T)$.

~~By Prop (Criteria p. 132), $\Rightarrow M\in\sigma(T)$.~~

3) $m\in\sigma(T)$. Consider $-T+MI$. Then $\sigma(-T+MI)\subseteq[0,M-m]$; hence $M-m\in\sigma(-T+MI)$ $\Rightarrow m\in\sigma(T)$.

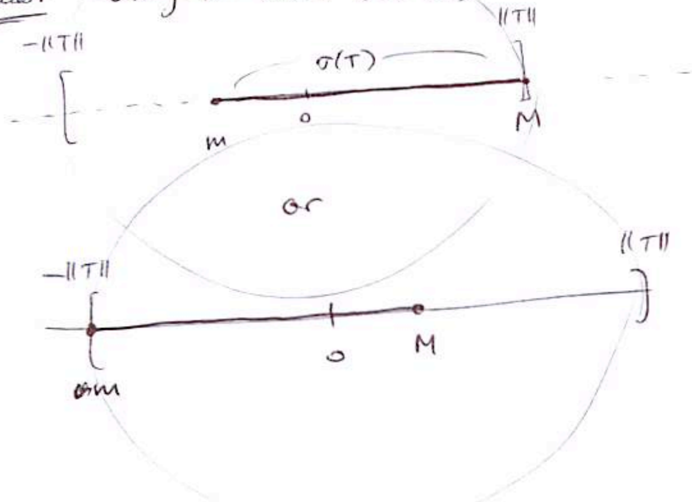
QED

(spectral radius)

Cor For selfadjoint $T \in \mathcal{L}(H,H)$,
$\|T\| \in \sigma(T)$ and in fact
$\|T\| = r(T) = \max\{|t| : t \in \sigma(T)\}$

Proof $\|T\| = \sup_{x \in S_H} |\langle Tx, x\rangle| = \max(|m|, |M|)$ in Thm p. 132
$= \max\{|t| : t \in \sigma(T)\}$ QED.

Remark ~~Only tho~~ looks like this:



Spectral theorem for compact selfadjoint operators

(eigenvectors)
Prop Let $T \in \mathcal{L}(H,H)$ be a selfadjoint operator. Then its eigenvectors corresponding ~~Let~~ to distinct eigenvalues are orthogonal.

Proof Let $Tx_1 = \lambda_1 x_1$, $Tx_2 = \lambda_2 x_2$. Then

$$\lambda_1 \langle x_1, x_2\rangle = \langle Tx_1, x_2\rangle = \langle x_1, Tx_2\rangle = \lambda_2 \langle x_1, x_2\rangle.$$

note: $\lambda_2 \in \mathbb{R}$ so $\bar{\lambda}_2 = \lambda_2$

Hence $\lambda_1 = \lambda_2$ or $\langle x_1, x_2\rangle = 0$

QED.

Def A subspace $E \subseteq H$ is called an invariant subspace of T if
$$T(E) \subseteq E.$$

- Examples: 1) The linear span of eigenvectors of T;
  2) Eigenspaces
- Invariant subspace problem: does every operator $T \in \mathcal{L}(\ell_2, \ell_2)$ have an invariant subspace?

Prop Let $T \in L(H,H)$ be selfadjoint.

If $E \subseteq H$ is an invariant subspace of T

then $E^\perp$ is also an invariant subspace of T.

Proof Let $x \in E^\perp$; we need to show that $Tx \in E^\perp$.

Let $y \in E$; then

$$\langle Tx, y\rangle = \langle x, \underbrace{Ty}_{\in E}\rangle = 0.$$ QED

~~**Lemma** Let $T \in L(H,H)$ be a compact self-adjoint operator, $H \neq \{0\}$.~~

~~Then T has eigen~~

~~Then T has nonzero eigenvectors.~~

THM (Spectral theorem for compact self-adjoint operators)

Let T be a compact self-adjoint linear operator on a separable Hilbert space H.

Then there exists an orthonormal basis (h_n) in H consisting of eigenvalues of T.

Proof 1) Let's first prove that T has a non-zero eigenvector.

Recall that since T is compact,

$$\sigma(T) = \sigma_p(T) \cup \{0\} \qquad (\text{Thm p.127})$$

If $\sigma_p(T) \neq \{0\}$ then T has nonzero eigenvectors.

If $\sigma_p(T) = \{0\}$ then by Cor p.134, $\|T\| = r(T) = 0 \Rightarrow T = 0$

$\Rightarrow$ every vector in H is an eigenvector of T.

2) Consider the family of all orthonormal sequences in H consisting of eigenvectors of T. By Zorn's Lemma, it has a maximal sequence (h_n). Let $E = \overline{\text{span}(h_n)}$. Suppose $E \neq H$. Clearly E is an invariant subspace of T; hence $E^\perp$ is also an invariant subspace (by Prop). Using Part 1 for the restriction of T onto $E^\perp$ we conclude that $E^\perp$ contains ~~an eigen~~ a nonzero eigenvector of T. But this contradicts the maximality of (h_n).

QED.

Remark (Diagonal Form)

- ~~Re~~. Since $Th_n = \lambda h_n$ for the orthonormal basis (h_n) of H, we have:

T is a diagonal operator in the o.n. basis of its eigenvectors.

- Recall that
$$x = \sum_{n=1}^{\infty} \langle x, h_n \rangle h_n \qquad , \text{ for } x \in H.$$

Then
$$Tx = \sum_{n=1}^{\infty} \langle x, h_n \rangle \underbrace{Th_n}_{= \lambda_n h_n}$$
$$\Downarrow$$
$$\boxed{Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, h_n \rangle h_n.} \qquad (*)$$

Remark (Non-separable spaces H.) (Ex).

- We never used (in a crucial way) that H is separable.
If H is a general Hilbert space, the argument above yields a similar version of THM for a possibly uncountable orthonormal basis (h_α) (ie $\langle h_\alpha, h_\beta \rangle = \delta_{\alpha,\beta}$; $\overline{\text{span}(h_\alpha)} = H$).

- Since there are only countably many h_α corresponding to nonzero eigenvalues λ_n (by compactness of T), one can deduce that

(*) still holds for ~~some~~ the orthonormal system (h_n) consisting of the ~~eigenvalues~~ eigenvectors corresponding to nonzero eigenvalues λ_n