Hilbert-Schmidt operators.

This is the most frequently used class of compact operators.

Def Let H be a separable Hilbert space with orthonormal basis $(e_i)_{i=1}^{\infty}$.
An operator $T \in L(H,H)$ is called a <u>Hilbert-Schmidt operator</u> if

$$\sum_{i=1}^{\infty} \|Te_i\|^2 < \infty$$

The quantity
$$\|T\|_{HS} = \left(\sum_{i=1}^{\infty} \|Te_i\|^2\right)^{1/2}$$
is called the Hilbert-Schmidt norm of T.

Prop The definition does not depend on the choice of the on basis (e_i).

Proof Assume that $\sum_i \|Te_i\|^2 < \infty$. ~~Let (e_i') be another orthonormal basis of H. Then~~
By Parseval's identity,

$$\sum_i \|Te_i\|^2 = \sum_{i,j} |\langle Te_i, e_j\rangle|^2 = \sum_{i,j} |\langle e_i, T^*e_j\rangle|^2 = \sum_j \|T^*e_j\|^2 \qquad (*)$$

Let $(\tilde{e}_i)$ be another orthonormal basis of H. Then, again using Parseval's identity,

$$\sum_j \|T^*e_j\|^2 = \sum_j \sum_i |\langle \tilde{e}_i, T^*e_j\rangle|^2 = \sum_i |\langle T\tilde{e}_i, e_j\rangle|^2 = \sum_i \|T\tilde{e}_i\|^2.$$

QED.

Remark We have also proved in $(*)$ that if T is Hilbert-Schmidt then so is T^*, and
$$\|T^*\|_{HS} = \|T\|.$$

Prop $\|T\| \le \|T\|_{HS}$

Proof Let $x = \sum_i a_i e_i \in H$. Then

$$\|Tx\| = \|T(\sum_i a_i e_i)\| = \|\sum_i a_i Te_i\| \le \sum_i |a_i| \cdot \|Te_i\| \quad (\text{by } \Delta)$$
$$\le \left(\sum_i |a_i|^2\right)^{1/2} \left(\sum_i \|Te_i\|^2\right)^{1/2} \quad (\text{by Cauchy-Schwartz})$$
$$\le \|x\| \cdot \|T\|_{HS}.$$

QED.

Examples: Multiplication op. on ℓ_2

$$T(x_i)_1^\infty = (\lambda_i x_i)_1^\infty$$

Since $Te_i = \lambda_i e_i$, ~~T is HS~~

$$\|T\|_{HS} = \left(\sum_i \lambda_i^2\right)^{1/2}.$$

$$\boxed{T \text{ is Hilbert-Schmidt iff } \sum_i \lambda_i^2 < \infty}$$

Compare: T is <u>Bounded</u> if $\sup_i |\lambda_i| < \infty$

Prop Hilbert-Schmidt operators are compact

Proof It suffices to show that a H.S operator T can be approximated by finite-rank operators (by Cor. p.109).

Define $T_n \in L(H,H)$ by

$$T_n e_i = \begin{cases} Te_i, & i \le n \\ 0, & i > n. \end{cases}$$

Then T has rank at most n, and

$$\|T - T_n\| \le \|T - T_n\|_{HS} = \left(\sum_{i>n} \|Te_i\|^2\right)^{1/2} \to 0 \quad \text{as } n \to \infty \qquad \text{QED.}$$

Applications to <u>Integral operators</u>

- A remarkable example of HS operators are <u>integral operators</u>.

Prop (<u>Integral operators are HS</u>).

Consider the integral operator T on $L_2(0,1)$ defined as

$$(Tf)(t) = \int_0^1 k(t,s)\, f(s)\, ds$$

with some ~~where the~~ kernel $k(t,s) \in L_2([0,1]^2)$.

Then T is a Hilbert-Schmidt operator and

$$\|T\|_{HS} = \|k\|_2 = \int_0^1\!\!\int_0^1 |k(t,s)|^2 \, ds\, dt$$

Remarks. 1). Formerly we proved that T is bounded ($\|T\| \le \|k\|_2$; p.65).

2). Compare to the <u>matrix</u> case: $\|T\|_{HS}^2 = \|(t_{ij})\|_2^2 = \sum |t_{ij}|^2$

Proof We look at the integral in the definition of $(Tf)(t)$ as an inner product.

let $k_t(s) = k(t,s)$; then

$$(Tf)(t) = \langle k_t, f \rangle \quad \text{for every } t \in [0,1]. \qquad (*)$$

Let $(e_i)_1^\infty$ be an orthonormal basis of $L_2[0,1]$. Then

$$\|T\|_{HS}^2 = \sum_i \|Te_i\|_{L_2}^2 = \sum_i \int_0^1 |(Te_i)(t)|^2 dt$$

$$= \sum_i \int_0^1 |\langle k_t, e_i \rangle|^2 dt = \int_0^1 \sum_i |\langle k_t, e_i \rangle|^2 dt$$

$$= \int_0^1 \|k_t\|_{L_2}^2 \, dt \quad \text{by Parseval}$$

$$= \int_0^1 \int_0^1 |k(t,s)|^2 \, ds \, dt.$$ QED.

- If the kernel is symmetric ($k(s,t) = \overline{k(t,s)}$) then the integral operator T is compact and self-adjoint.

Therefore, we can apply the Spectral theorem to study it.

This theorem yields an orthonormal basis (e_i) of eigenvectors of T.

Being an eigenvector means that

$$(Te_i)(t) = \int_0^1 k(t,s) \, e_i(s) \, ds = \lambda_i e_i(t).$$

where λ_i is the corresponding eigenvalue.

Writing this as in $(*)$, $\langle k_t, e_i \rangle = \lambda_i e_i(t)$,

~~we obtain~~ and substituting in the orthogonal expansion,

$$k_t(s) = \sum_i \langle k_t, e_i \rangle e_i(s)$$

we obtain

$$\boxed{k(t,s) = \sum_i \lambda_i \overline{e_i(t)} \, e_i(s)} \qquad (*)$$

(Variables separated!)

- This means that, for every fixed t, the series converges in $L_2[0,1]$ (w.r.to s).

A stronger result is true - convergence holds in $L_2([0,1]^2)$:

THM (Hilbert - Schmidt)

Consider a function $k(t,s) \in L_2([0,1]^2)$ such that $k(t,s) = k(s,t)$.

There exists an orthonormal basis $(e_i)_1^\infty$ of $L_2[0,1]$ ~~such~~

and numbers $\lambda_i \in \mathbb{R}$ such that

$$k(t,s) = \sum_i \lambda_i e_i(t) e_i(s).$$

The convergence is understood in $L_2([0,1]^2)$.

Moreover, e_i are the eigenvectors of the integral operator with kernel $k(t,s)$, corresponding to nonzero eigenvalues λ_i.

Proof The functions

$$h_{ij}(t,s) = e_i(t) e_j(s)$$

form an orthonormal basis of $L_2([0,1]^2)$ (Ex: Check!)

Hence we can expand

$$k = \sum_{i,j} \langle k, h_{ij} \rangle h_{ij}.$$

$$\langle k, h_{ij} \rangle = \int_0^1 \int_0^1 k(t,s) e_i(t) e_j(s) \, dt \, ds$$

$$= \int_0^1 \underbrace{\left(\int_0^1 k(t,s) e_j(s) \, ds \right)}_{(Te_j)(t)} e_i(t) \, dt$$

$$= \langle Te_j, e_i \rangle = \langle \lambda_j e_j, e_i \rangle = \begin{cases} \lambda_i & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Hence $k = \sum_i \lambda_i h_{ii} = \sum_i \lambda_i e_i(t) e_i(s)$. QED.